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# ON THE RATES OF CONVERGENCE OF FULLY DISCRETE SOLUTIONS OF HAMILTON-JACOBI EQUATIONS

## ESTIMATIONS DES VITESSES DE CONVERGENCE DES SOLUTIONS APPROCHEES DES EQUATIONS D'HAMILTON-JACOBI

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## ABSTRACT

In this paper we consider the Hamilton-Jacobi equation associated to an infinite horizon optimal deterministic control problem. We consider approximated solutions by discretization on time and spacial variables and we study the rate of convergence of the approximate solution to the optimal cost function of the original problem. We give several explicit estimates of the rate of convergence which depend on the regularity of the data and on the parameters of discretization. Finally we present some examples which show that these estimates are optimal.



## RESUME

On considère ici l'équation de Hamilton-Jacobi associée à un problème de contrôle optimal avec horizon infini. On étudie les solutiones approchées obtenues par discretisation dans les variables d'espace et de temps et la vitesse de convergence des solutiones approchées vers la fonction de coût optimale. On donne des plusieurs estimations explicites de la vitesse de convergence qui dependent de la régularité des données du problème et des paramètres de discretisation. Finalment on présente quelques exemples qui montrent que ces estimationes sont optimales.



*Key words and phrases.* Hamilton-Jacobi equation, numerical approximations, error estimates, optimal estimates.

CONTENTS

*What is now proved, was once only imagined*

William Blake

The Marriage of Heaven and Hell

## INTRODUCTION

In this paper we study the Hamilton-Jacobi equation associated to an infinite horizon optimal deterministic control problem. We consider approximated solutions by discretization on time and spacial variables by means of finite elements techniques. We use the discretization procedures described in [3], [4], [5], [17].

Our aim is to obtain explicit estimates of the rate of convergence of the approximate solution to the optimal cost function of the original problem. Previous results in this direction have been obtained in [5], [11], [17]. Here, we have tried to get the optimal estimates depending on the regularity of the problem and on the parameters of discretization h, k (time and space discretizaton steps). In each case we have sought examples that show the optimality of the corresponding estimates. Also, we have studied the optimization of parameters h, k in order to obtain the best possible results available with a methodology of this type.

The principal results are the following:

- Theorem 3.1 gives an estimate of the error of the fully discrete solution as a function of parameters h, k.
- This estimate indicates that for the general case, when the optimal cost function is only Lipschitz continuous (for which heavy assumptions about the discount factor $\lambda$ are necessary), the best resut one can expect is a convergence of order $k^{1/2}$.
- Also, in the general case with a small discount coefficient $\lambda$, (where u is only Hölder continuous of order $\gamma < 1$ ) the best convergence rate one can expect to get is of order $k^{\gamma/2}$.
- We have obtained the optimal relation necessary to hold between the parameters h and k, in order to get the minimum error of discretization. The optimal value of h is different provided we deal with the general case or with the regular case. In the general case the optimal convergence rate is of order $k^{\gamma/2}$, (provided a step $h \leq k$ is used). In the regular case, where semiconcavity assumptions and the condition $\gamma > 1$ imply that the errors of the time discretization procedure are only of order h (not of order $h^{1/2}$, as it happens in the general case), the estimate and of course the numerical results can be improved. This occurs when an unusual large time step $(\bar{h} = k^{\eta})$ is employed, with $\eta = (2/(\gamma+1) \wedge (2/3))$. The total error in this case is of order $k^{\xi}$, with $\xi = (\gamma/(\gamma+1)) \wedge (2/3)$.

## 1. GENERAL DESCRIPTION OF THE PROBLEM

### 1.1 The Optimal Cost Function u

Let u be the optimal cost function associated to an infinite horizon deterministic control problem in the following form:

$$u(x) = \inf_{a \in \mathcal{A}} J(x,\ a(\cdot)) \tag{1}$$

where

$$J(x, a(\cdot)) = \int_0^\infty f(y(t), a(t)) e^{-\lambda t}\, dt$$

Here $\mathcal{A}$ denotes the set of all measurable functions of $[0, +\infty)$ to a given compact subset $A$ of $\Re^n$ i.e. $\mathcal{A} = \{\ a{:}[0,\infty) \to A,\ a(\cdot) \text{ measurable } \}$.

The state of the system is given by the following differential equation.

$$\dot{y}(t) = g(y(t), a(t)) \tag{2}$$

$$y(0) = x, x \in \Omega, \Omega \subset \Re^\nu, \text{ open.}$$

The mapping $y{:}\ \Re^\nu \text{x } \mathcal{A} \to \Re^\nu$ is called the *response* or the state corresponding to the control $a(\cdot)$. We suppose that $\forall\ t > 0$, $y(t) \in \Omega$, independently of the applied control. The constant $\lambda$ is the *discount factor* and the function f is the *instantaneous cost.*

We assume f and g satisfies:

$$\left| \begin{array}{l} \| g(x,a) - g(\tilde{x},a) \| \le L_g \| x - \tilde{x} \|, \\ \\ \| g(x,a) \| \le M_g \end{array} \right. \tag{3}$$

$$\left| \begin{array}{l} | f(x,a) - f(\tilde{x},a) | \le L_f \| x - \tilde{x} \|, \\ \\ | f(x,a) | \le M_f \end{array} \right. \tag{4}$$

We will denote with $H_1$ the set of inequalities (3) and (4) and we will impose its validity along the rest of the paper.

### 1.2 Properties of u

It is well known (see [8], [22]) that under the assumptions above made, the value function u is the unique bounded uniformly continuous viscosity solution of the Hamilton-Jacobi-Bellman equation:

$$\boxed{\min_{a\in A}\left(\frac{\partial u}{\partial x}(x)\ g(x,a)) - \lambda u(x) + f(x,a)\right) = 0} \tag{5}$$

and that it satisfies (see [3], [4]):

$$|u(x)| \le \frac{M_f}{\lambda}, \tag{6}$$

$$|u(x) - u(\hat{x})| \le L_u \| x - \hat{x} \|^{\gamma}, \ \forall x \in \Omega, \tag{7}$$

where

$$L_u = \begin{cases} \dfrac{L_f}{\lambda - L_g} & \gamma = 1 \quad \text{if } \lambda > Lg, \\ L_f^{\gamma}(2M_f)^{1-\gamma}\left(\frac{1}{\lambda} + \frac{1}{Lg - \lambda}\right), & \gamma = \frac{\lambda}{L_g} \quad \text{if } \lambda < L_g, \\ \left(\frac{2M_f}{L_f}\right)^{1-\gamma} \frac{1}{1-\gamma} e^{-\gamma}, & \gamma \in (0,1) \quad \text{if } \lambda = L_g. \end{cases} \tag{8}$$

i.e. u is Lipschitz continuous under assumption $\lambda > L_g$ and only Hölder continuous if $\lambda \le L_g$ (see [4], [13], [14], [17]).

## 2. NUMERICAL SOLUTION

### 2.1 A Discrete Time Approximation

The first step to get a discrete problem amenable with numerical methods is to do a discretization in the time variable. This procedure also has good properties for the aim to obtain sub-optimal feedback policies (see [3]).

Let $h > 0$ and consider the Hamilton-Jacobi-Bellman equation discretized on time.

$$\boxed{u^h(x) = \min_{a\in A}\left((1 - \lambda h)\, u^h(x + h\, g(x,a)) + h\, f(x,a)\right)} \tag{9}$$

#### 2.1.1 *Properties of* $u^h(x)$

It has been proved in [4] that (9) has a unique solution given by the unique fixed point of operator $A^h$, where:

$$A^h : C(\Omega) \to C(\Omega)$$
$$(A^h w)(x) = \min_{a \in A} ((1-\lambda h) w(x + h g(x,a)) + h f(x,a)) \tag{10}$$

In order to have a well defined operator $A^h$, we will suppose that:

$$x + h g(x,a) \in \Omega, \quad \forall x \in \Omega, \quad \forall a \in A, \quad \forall h \leq h_0 < \frac{1}{\lambda} \tag{11}$$

Moreover we define:

$$A_a^h : C(\Omega) \to C(\Omega)$$
$$(A_a^h w)(x) = ((1-\lambda h) w(x + h g(x,a)) + h f(x,a)) \tag{12}$$

trivially we have

$$(A^h w)(x) \leq (A_a^h w)(x) \ \forall \ a \in A \tag{13}$$

For $u^h$, the following representation formula holds (see [4]):

$$u^h(x) = \inf_{a \in \mathcal{A}^h} J^h(x,a) \quad \forall \ x \in \Omega \tag{14}$$

where:

$$\mathcal{A}^h = \left\{ a(jh) / a(jh) \in A, \ j = 1, \dots \right\}$$

and

$$J^h(x,a) = h \sum_{j=1}^{\infty} f(y^h(x,j)), a(jh))(1 - \lambda h)^{j-1} \tag{15}$$

In (15) the sequence $y^h(x,j)$ is determined by the recursion:

$$y^h(x, j+1) = y^h(x,j) + h g(y^h(x,j), a((j+1)h)), \ j = 0, \dots \tag{16}$$

with

$$y^h(x,0) = x$$

$u^h$ verifies the following properties (see [3] )

$$\left|u^h(x)\right| \leq \frac{M_f}{\lambda}, \quad \left|u^h(x) - u^h(\hat{x})\right| \leq L_{u^h} \| x - \hat{x} \|^{\gamma} \tag{17}$$

$\forall\ h \in (0, \frac{1}{\lambda}),\ \forall x, \hat{x} \in \Omega$, where:

$$L_{u^h} = \begin{cases} \dfrac{L_f}{\lambda - L_g} & \gamma = 1 \quad \text{if } \lambda > L_g, \\[2ex] L_f^{\gamma}\,(2M_f)^{1-\gamma}\left(\frac{1}{\lambda} + \frac{1}{L_g - \lambda}\right) & \gamma = \frac{\lambda}{L_g} \quad \text{if } \lambda < L_g, \\[2ex] \left(\frac{2\,M_f}{L_f}\right)^{1-\gamma} \frac{1}{1-\gamma} e^{-\gamma}, & \gamma \in (0,1) \quad \text{if } \lambda = L_g. \end{cases} \tag{18}$$

### 2.1.2 *Rate of convergence*

The convergence of discrete time solution $u^h$ to the continuous solution u is uniform; indeed, it is possible to establish explicit estimates of the rate of convergence.

Theorem 2.1: *Under assumptions* $H_1$ *the following estimates holds:*

$$\left|u(x) - u^h(x)\right| \leq C h^{\gamma/2} \tag{19}$$

*where,* $\gamma = 1$ *if* $\lambda > L_g$, $\gamma = \frac{\lambda}{L_g}$ *if* $\lambda < L_g$, *and* $\gamma$ *is an arbitrary number in* $(0,1)$ *if* $\lambda = L_g$.

*Moreover, under the following additional hypotheses of semiconcavity of* g, f:

$$\| g(x - \hat{x}) - 2g(x) + g(x + \hat{x}) \| \leq C \| \hat{x} \|^2 \tag{20}$$

$$\left| f(x - \hat{x}) - 2f(x) + f(x + \hat{x}) \right| \leq C \| \hat{x} \|^2 \tag{21}$$

*the estimate*

$$\left|u(x) - u^h(x)\right| \leq C h^{\gamma/2 \wedge 1}$$

*is valid for all* $\gamma$.

*Which implies in addition that if it is satisfied the condition*

$$\lambda > 2L_g, \tag{22}$$

*the following estimate, sharper than* (19), *holds*

$$\left| u(x) - u^h(x) \right| \leq Ch \tag{23}$$

**Remark 2.1:** In (19), when $\gamma = L_g$ the value of C depends on the chosen value of $\gamma$.

**Remark 2.2:** The proof of these results will be omited here. We will refer the reader to the proofs contained in [3], [20]. They are based in techniques of convex analysis [25] and classical methods usually employed in the treatment of viscosity solution [22].

**Note:** From here, and in order to obtain simplicity of notation and clarity of arguments, we will use letters C, M, K to denote arbitrary constants (which values depends on the context where they appear) which depend *on the data of the problem (constants* $\lambda$, $M_g$, $M_f$, $L_g$, $L_f$, *etc.) but do not depend on the* parameters of discretizations h, k, etc.

## 2.2 Fully Discrete Solution

The discretization above introduced remains a theoretical one. To obtain computational results it is also necessary to do a discretization of space variables and to give a numerical method to compute $u^h$. This is done here via a discretization in the state variables by means of finite elements techniques.

### 2.2.1 *Description of the problem*

a) Approximation of domain $\Omega$

We will identify the discretization in space variable with the parameter k, which also indicate the size of the discretization.

Let $\Omega$ be an open set of $\Re^\nu$ and $\{S_j^k\}$ a family of regular triangulations of $\Omega$, i.e. a finite sets of simplices (see [15],[16], [23]) that approaches $\Omega$ in the following sense:

$\Omega_k = \bigcup_j S_j^k$ is a polyhedron of $\Re^\nu$ *such that the following properties are satisfied:*

- $$\max_j (\text{diám } S_j^k) = k \tag{24}$$

- $$\exists\ h_0 > 0 \text{ such that } x + hg(x,a) \in \Omega_k \quad \forall x \in \Omega_k,\ \forall\ a \in A,\ \forall\ h < h_0 \tag{25}$$

- $\Omega_k \to \Omega$, when $k \to 0$, in the following sense:

$$\forall \text{ K compact contained in } \Omega,\ \exists\ \bar{k}(K) \ /\ K \subset \Omega_k \quad \forall\ k < \bar{k}(K) \tag{26}$$

- If $d_i$ is the diameter of the simplex $S_i^k$, then: $\exists\ X_1 > 0$ such that for any simplex of $\Omega_k$, there exists a sphere of radius

$$r \geq X_1 d_i \tag{27}$$

in the interior of the simplex.

- Moreover there exists M, independent of the discretization such that:

$$\frac{k}{d_i} \leq M \ \forall i. \tag{28}$$

b) Definition of the approximation space $W_k$

We consider the set $W_k$ of functions $w: \Omega_k \to \Re$, w continuous in $\Omega_k$, $\frac{\partial w}{\partial x}$ constant in the interior of each simplex of $\Omega_k$ (i.e. w are linear finite elements)

Note: If denote with N the cardinality of set of vertices of $\Omega_k$, being also $\{x_i, i = 1, \ldots, N\}$ an apropriate order in the set of nodes (vertices of simplex) of $\Omega_k$, it is obvious that any $w \in W_k$ is completely characterized by the values $w(x_i), \forall i = 1,\ldots, N$.

c) Definition of operator $A_k^h$

We define operator $A_k^h$ in the following way:

$$A_k^h : W_k \to W_k$$

$$(A_k^h w)(x_i) = \min_{a \in A} \Big( (1 - \lambda h) w(x_i + h g(x_i, a)) + h f(x_i, a) \Big) \qquad \forall\, i=1,\ldots,N \tag{29}$$

d) Fully discrete problem:

The final problem, amenable to be solved numerically is the following:

$$\boxed{\text{Problem } P_k\text{: } \textit{Find the fixed point of operator } A_k^h} \tag{30}$$

Note: It is clear that if A is an infinite set, problem $P_k$ is not actually a fully discrete problem. In order to obtain a problem with those characteristics, a final step of discretization remains to be done: ***the discretization of the set A by a finite set*** $A_k^h$. This procedure introduces another error of discretization which size can be bounded by $d(A^k, A)/\lambda$. In this paper, we will not deal with this problem, which essentially only introduces a cumbersome notation; so, we will suppose from here that either $A$ is finite or that the whole set $A$ is considered in the definition of $A_k^h$.

e) An equivalent formulation

As every point x of $\Omega_k$ is a convex combination of vertices of the simplices to which it belongs, then $\forall$ $a \in A$ $\exists$ a matrix $\Lambda(a) \in \Re^{N\mathrm{x}N}$ with entries $\alpha_j(x_i,a)$, such that the following properties hold:

$$\alpha_j(x_i,a) \geq 0 \ \forall i,j \qquad \sum_{j=1}^{N} \alpha_j(x_i,a) = 1, \ \forall i \qquad x_i + h\,g(x_i,a) = \sum_{j=1}^{N} \alpha_j(x_i,a)x_j \ \forall i \tag{31}$$

Hence:

$$(A_k^h w)(x_i) = \min_{a \in A} \left( (1-\lambda h) \sum_{j=1}^{N} \alpha_j(x_i,a)w(x_j) + h\,f(x_i,a) \right) \tag{32}$$

We have the following relation between $w \in W_k$ and vectors $U \in \Re^N$,

$$w(x_i) \Leftrightarrow u_i$$

and the corresponding fixed point problem has the equivalent form:

$$U = \min_{a \in A} \left( (1-\lambda h)\ \Lambda(a)U + hF(a) \right) \tag{33}$$

where $F(a) \in \Re^N$ is defined by $F_i(a) = f(x_i,a)$ i=1, ... N, and $\Lambda(a)$ is the matrix with entries $\Lambda(a)_{i,j} = \alpha_j(x_i,a)$

Remark 2.3: By virtue of $A_k^h$ definition, we have, for any $w \in C(\Omega)$:

$$(A_k^h \tilde{w})(x_i) = \min_{a \in A} \Big( (1 - \lambda h) \sum_{j=1}^{N} \alpha_j(x_i, a)\, \tilde{w}(x_j) + h f(x_i, a) \Big) = (A^h \tilde{w})(x_i) \tag{34}$$

with $\tilde{w}$ the linear interpolation of w in each simplex of $\Omega_k$, which coincides with w in any vertex of the triangulation $\Omega_k$.

### 2.2.2 *Existence and uniqueness of the fully discrete solution*

Theorem 2.2: *There exists a unique solution* $u_k^h$ *of problem* $P_k$.

Proof: We introduce in $W_k$ the norm:

$$\| w \| = \max_i \big| w(x_i) \big|$$

It is clear that $A_k^h$ is a contractive operator, i.e.

$$\Big\| A_k^h(w) - A_k^h(\hat{w}) \Big\| \le (1 - \lambda h) \| w - \hat{w} \|, \quad \forall\, w, \hat{w} \in W_k \tag{35}$$

and then it has a unique fixed point. If we denote it with $u_k^h$, we have:

$$u_k^h = A_k^h(u_k^h) \tag{36}$$

□

Remark 2.4: Problem $P_k$ is a nonlinear fixed point problem with a very special structure. It corresponds or is equivalent to an optimal stochastic control problem on a Markov Chain. (The reason of this relation is the existence of a discrete maximum principle (see [6], [7]) valid for the schemes of discretization used in the definition of $A_k^h$ ). So, it can be solved with any of the several methods of Picard or Howard type described in the current literature (see [2], [9], [10], [21], [26],[27]).

In this paper we will neither deal, with the computational procedure to solve ((30) ) nor with any application but let us say that in the real application of the methodology theoretically studied in this paper (e.g. [1]) we have used the special techniques with acceleration procedures studied in [18], [19].

2.2.3 *An equivalent problem*

We will see that, in a broad sense, problem $P_k$ is equivalent to problem $\hat{P}_k$ where $\hat{P}_k$ consists in to find the fixed point of $\hat{A}_k^h$, being

$$(\hat{A}_k^h w)(x_i) = \min_{a \in A} \left( \frac{1}{1+\lambda h} w(x_i + h g(x_i, a)) + h f(x_i, a) \right) \qquad \forall\, i=1,\ldots,N \tag{37}$$

In a strict form, the following estimate holds:

$$\left\| u_k^h - \hat{u}_k^h \right\| \le h M_f \tag{38}$$

In effect:

$$u_k^h(x) - \hat{u}_k^h(x) \le (1-\lambda h) u_k^h(x + h g(x,\bar{a})) - \frac{1}{1+\lambda h} \hat{u}_k^h(x + h g(x,\bar{a})) \tag{39}$$

where $\bar{a}$ realizes the minimum in (37). As

$$\left\| \hat{u}_k^h \right\| \le \frac{1+\lambda h}{\lambda} M_f$$

(39) becomes:

$$u_k^h(x) - \hat{u}_k^h(x) \le (1-\lambda h) \left\| u_k^h - \hat{u}_k^h \right\| + \left( \frac{1}{1+\lambda h} - (1-\lambda h) \right) \frac{1+\lambda h}{\lambda} M_f \tag{40}$$

Analogously we can obtain the estimate

$$\hat{u}_k^h(x) - u_k^h(x) \le (1-\lambda h) \left\| u_k^h - \hat{u}_k^h \right\| + \left( \frac{1}{1+\lambda h} - (1-\lambda h) \right) \frac{1+\lambda h}{\lambda} M_f \tag{41}$$

then by virtue of (40) and (41) , we have:

$$\left\| u_k^h - \hat{u}_k^h \right\| \le h M_f$$

**Note:** As it is obvious, no restrictions (of type (11)) on the value of h are necessary to hold in order to have a well defined problem $\hat{P}_k$.

## 3. CONVERGENCE OF FULLY DISCRETE PROBLEM

### 3.0 Presentation of the Central Result

In order to keep the employed ideas clear and to simplify the exposition of the arguments we will deal only with the case $\Omega = \Re^n$. The general case can be treated without fundamental changes using the perturbation of domain techniques described in [17]. The main result is:

Theorem 3.1: *Assume* (11), $H_1$ *and the general properties of triangulation above described in* §2.2.1 *hold. Then, there is a constant* M *independent of* h, k *such that*

$$\left| u(x) - u_k^h(x) \right| \leq \begin{cases} M(\sqrt{h} + \frac{k}{\sqrt{h}}), & \text{if } L_g < \lambda \\ M(\sqrt{h} + \frac{k}{\sqrt{h}})^{\gamma}, \quad \text{with } \gamma = \frac{\lambda}{L_g} & \text{if } L_g > \lambda \\ M(\sqrt{h} + \frac{k}{\sqrt{h}})^{\gamma}, \quad \gamma \in (0,1) & \text{if } L_g = \lambda \end{cases} \tag{42}$$

*Also, if* (20) *and* (21) *are valid then the above estimations can be improved, i.e. it holds that:*

$$\left| u(x) - u_k^h(x) \right| \leq \begin{cases} M(h + \frac{k}{\sqrt{h}}), & \text{if } 2L_g < \lambda \\ M(h^{\gamma} + \frac{k}{\sqrt{h}}), \quad \text{with } \gamma \in (0,1) & \text{if } 2L_g = \lambda \\ M(h^{\gamma/2} + \frac{k}{\sqrt{h}}), \quad \text{with } \gamma = \frac{\lambda}{L_g} & \text{if } L_g < \lambda < 2L_g \\ M\left(h^{\gamma/2} + \left(\frac{k}{\sqrt{h}}\right)^{\gamma}\right), \quad \text{with } \gamma \in (0,1) & \text{if } L_g = \lambda \\ M\left(h^{\gamma/2} + \left(\frac{k}{\sqrt{h}}\right)^{\gamma}\right), \quad \text{with } \gamma = \frac{\lambda}{L_g} & \text{if } L_g > \lambda \end{cases} \tag{43}$$

To achieve these results we need to prove some previous properties.

### 3.1 Introduction of Auxiliary Problems

#### 3.1.1 *Definition of $u_T$, optimal cost function for a problem with finite horizon*

A key point of the proof of the explicit error estimates (42) and (43) is the introduction of the finite time optimization problem:

$$\inf_{a\in\mathcal{A}} \int_0^T f(y(t,a(t)))e^{-\lambda t}\,dt, \quad \text{where } y(\cdot) \text{ verifies (2)}$$

and the use of the associated optimal cost function $u_T$ and its properties, with

$$u_T(x) = \inf_{a\in\mathcal{A}} \int_0^T f(y(t,a(t)))e^{-\lambda t}\,dt \tag{44}$$

#### 3.1.2 *Definition of $u_n^h$, discrete time optimal cost function for a problem with finite horizon*

We define recursively for $0 \le n \le \mu = \frac{T}{h}$, the optimal cost function $u_n^h(x_i)$ associated to the optimization problem with finite horizon and step control functions corresponding to the discretization on time of problem (44). We proceed in the following way:

$$\begin{aligned} u_{n-1}^h &= A^h u_n^h \\ u_\mu^h &= 0 \end{aligned} \tag{45}$$

and we will denote $u_T^h = u_0^h$.

Note: In order to simplify the analysis of auxiliary problems and the proof of convergence properties, we will suppose, w.l.g. that $\frac{T}{h}$ is always an integer.

It is easy to see that:

$$u_T^h(x) = h \sum_{j=0}^{\mu-1} f(y^h(x,j),\bar{a}(jh))(1-\lambda h)^j, \text{ with } T = \mu h \tag{46}$$

where $\bar{a}(jh)$ is the control that realizes the minimum in (10), when $A^h$ is applied to function $u_j^h$.

**Remark 3.1:** It must be noted that $\bar{a}(jh)$ depends on the initial position of the system x and it is defined recursively by the relation:

$$u^h_{n-1}(y^h(x,n-1)) = (A^h u^h_n)(y^h(x,n-1)) =$$

$$= \min_{a\in A}\Big((1-\lambda h)u^h_n(y^h(x,n-1)+hg(y^h(x,n-1),a))+hf(y^h(x,n-1),a)\Big) =$$

$$= (1-\lambda h)u^h_n(y^h(x,n-1)+hg(y^h(x,n-1),\bar{a}((n-1)h))+$$

$$+hf(y^h(x,n-1),\bar{a}((n-1)h))) \tag{47}$$

where the sequence $y^h(x,j)$ is determined by the recursion (16).

### 3.1.3 *Definition of $u^h_{k,n}$, fully discrete optimal cost function for a problem with finite horizon*

We define recursively for $1 \le n \le \mu = \frac{T}{h}$ function $u^h_{k,n}$, fully discrete solution of the problem with finite horizon, in the following way:

$$u^h_{k,n-1} = A^h_k u^h_{k,n} \tag{48}$$

$$u^h_{k,\mu} = 0$$

and we will denote $u^h_{k,T} = u^h_{k,0}$.

### 3.2 Properties of Auxiliary Functions:

The procedure applied to prove the estimate of $\left|u(x) - u^h_k(x)\right|$, consists in the decomposition of it in the following four successive estimates:

$$\left|u(x)-u^h_k(x)\right| \le \left|u(x)-u_T(x)\right| + \left|u_T(x)-u^h_T(x)\right| +$$

$$+\left|u^h_T(x)-u^h_{k,T}(x)\right| + \left|u^h_{k,T}(x)-u^h_k(x)\right| \tag{49}$$

Estimates for the right terms are given by the properties established in the following lemmas:

**Lemma 3.1:** *Under assumptions* $H_1$*, we have:*

$$\left| u(x) - u_T(x) \right| \le \frac{M_f}{\lambda} e^{-\lambda T} \tag{50}$$

**Lemma 3.2:** *Under assumptions* $H_1$ *and* (11)*, we have:*

$$\left| u_T(x) - u_T^h(x) \right| \le \begin{cases} C\sqrt{h} & \text{if } L_g < \lambda \\ C\sqrt{h}\, e^{(L_g-\lambda)T} & \text{if } L_g > \lambda \\ C\sqrt{h}\, T & \text{if } L_g = \lambda \end{cases} \tag{51}$$

*Also if* (20) *and* (21) *are valid, then the above estimations can be improved; i.e. it holds that:*

$$\left| u_T(x) - u_T^h(x) \right| \le \begin{cases} C h & \text{if } 2L_g < \lambda \\ C h\, e^{(2L_g-\lambda)T} & \text{if } 2L_g > \lambda \\ C h T & \text{if } 2L_g = \lambda \end{cases} \tag{52}$$

**Lemma 3.3:** *Let* $H_1$ *and* (11) *be satisfied. Then:*

$$\left| u_{k,T}^h(x) - u_k^h(x) \right| \le \frac{M_f}{\lambda} (1 - \lambda h)^{\mu} \le \frac{M_f}{\lambda} e^{-\lambda T} \tag{53}$$

**Lemma 3.4:** *Under assumptions* $H_1$, (11), (27) *and* (28) *we have:*

$$\left| u_T^h(x) - u_{k,T}^h(x) \right| \le \begin{cases} M \frac{k}{\sqrt{h}} & \text{if } L_g < \lambda \\ M e^{(L_g - \lambda)T} \frac{k}{\sqrt{h}} & \text{if } L_g > \lambda \\ M T \frac{k}{\sqrt{h}} & \text{if } L_g = \lambda \end{cases} \tag{54}$$

### 3.3 Proof of Theorem 3.1

Proof: In order to simplify the proof of this theorem, we will refer the reader to the appendix to find there the proofs of Lemmas 3.1, 3.3 and 3.4; the proof of Lemma 3.2 is contained in [3], [20].

By virtue of Lemmas 3.1 to 3.4, we have that the analysis of (49) is reduced to deal with the following three cases:

- **Case $L_g > \lambda$**

In this case (50), (51), (53) and (54) give:

$$\left|u(x)-u_k^h(x)\right| \le \left|u(x)-u_T(x)\right| + \left|u_T(x)-u_T^h(x)\right| +$$

$$+\left|u_T^h(x)-u_{k,T}^h(x)\right| + \left|u_{k,T}^h(x)-u_k^h(x)\right| \le$$

$$\le \frac{M_f}{\lambda}e^{-\lambda T} + C\sqrt{h}\,e^{(Lg-\lambda)T} + M\,e^{(Lg-\lambda)T}\frac{k}{\sqrt{h}} + \frac{M_f}{\lambda}e^{-\lambda T} \tag{55}$$

and we obtain:

$$\left|u(x)-u_k^h(x)\right| \le M_1\left(e^{-\lambda T} + e^{(Lg-\lambda)T}\left(\sqrt{h}+\frac{k}{\sqrt{h}}\right)\right) \tag{56}$$

where

$$M_1 = \max\left(2\frac{M_f}{\lambda}, C, M\right)$$

Minimizing (56) in T, we obtain:

$$\left|u(x)-u_k^h(x)\right| \le K\left(\sqrt{h}+\frac{k}{\sqrt{h}}\right)^{\gamma}, \tag{57}$$

where

$$K = M_1\left(\left(\frac{Lg-\lambda}{\lambda}\right)^{\frac{\lambda}{Lg}} + \left(\frac{Lg-\lambda}{\lambda}\right)^{\frac{\lambda}{Lg}-1}\right)$$

with

$$\gamma = \frac{\lambda}{Lg}$$

Note: Indeed, the minimum in (56) should be computed on the discrete set $\{nh/n=0,1,\ldots\}$; if we do the minimization procedure on the set $[0, \infty)$, we obtain an equivalent result up to an additional term of order h, that does not essentially modifies the estimate (57).

• Case $L_g < \lambda$

In this case, by virtue of (51), (55) becomes:

$$\left| u(x) - u_k(x) \right| \leq \frac{M_f}{\lambda} e^{-\lambda T} + C\sqrt{h} + M \frac{k}{\sqrt{h}} + \frac{M_f}{\lambda} e^{-\lambda T}$$

then, by taking the limite $T \rightarrow \infty$, we obtain

$$\left| u(x) - u_k^h(x) \right| \leq M_1 \left( \sqrt{h} + \frac{k}{\sqrt{h}} \right) \tag{58}$$

where $M_1 = \max(C, M)$

• Case $L_g = \lambda$

In this case, by virtue of (51) and (54), (55) becomes:

$$\left| u(x) - u_k^h(x) \right| \leq \frac{M_f}{\lambda} e^{-\lambda T} + C\sqrt{h}\, T + M \frac{k}{\sqrt{h}} T + \frac{M_f}{\lambda} e^{-\lambda T}$$

and we have

$$\left| u(x) - u_k^h(x) \right| \leq M_1 \left( e^{-\lambda T} + (\sqrt{h} + \frac{k}{\sqrt{h}}) T \right) \tag{59}$$

where $M_1 = \max\left( 2\frac{M_f}{\lambda}, C, M \right)$

For $\frac{1}{\lambda}(\sqrt{h} + \frac{k}{\sqrt{h}}) \leq 1$, the minimum of the rigth hand side of (59) is realized by

$$\bar{T} = -\frac{1}{\lambda} \ln \frac{1}{\lambda}(\sqrt{h} + \frac{k}{\sqrt{h}})$$

so, replacing it in (59), we have:

$$\left| u(x) - u_k^h(x) \right| \leq M_1 \left( h + \frac{1}{\lambda}(h + \frac{k}{\sqrt{h}}) - \frac{1}{\lambda}(h + \frac{k}{\sqrt{h}}) \ln \frac{1}{\lambda}(h + \frac{k}{\sqrt{h}}) \right)$$

but, as the following property holds

$$-x \ln x \leq K\, x^{\gamma},\ \gamma \in (0,1),\ K = \frac{1}{1-\gamma}\, e^{-1},$$

then we have:

$$\left| u(x) - u_k^h(x) \right| \leq C \left( \sqrt{h} + \frac{k}{\sqrt{h}} \right)^{\gamma}$$

where $C = \frac{M_1}{\lambda} \max \left(1, \frac{1}{1-\gamma}\, e^{-1}\right)$

The estimate (43) can be proved in a similar way, taking in mind (52)

□

### 3.4 Discusion. Relation between h and k

Estimates (42), (43) give a general expression of the error estimates, depending on both parameters h, k. From them, two issues arise immediatly:

- the optimal relation between h and k.
- the optimality of the estimates.

The following paragraphs deal with both issues.

#### 3.4.1 *Optimality studies*

Theorem 3.1 gives an estimate of the error of the fully discrete solution as a function of both parameters $h, k$. The parameter k is related to the size of the triangulation $\Omega_k$ and to the cardinality N of the mesh; obviously, it measures the quantity of information to be processed. Once this is realized, it arises the problem of parameter h optimization, i.e. to determine the value of h that brings the least total error considering that a mesh of fixed size k is given. This optimal value of h is found minimizing the expression (42) and (43); the answer is different provided we deal with the general case (where only (42) is valid) or the regular case, where the semiconcavity assumptions imply the sharper estimate (43).

a) General case

Here, only the estimate (42) holds. If we consider the minimization in h, we obtain that $\bar{h} = k$ realizes the minimum in (42). This result coincides with the usual case, because generally it is taken a time step h of the same order of k, i.e. there exists $m_1$, $m_2$ such that:

$$m_1 k \leq h \leq m_2 k$$

In this case the estimate (42) becomes

$$\left| u(x) - u_k^h(x) \right| \leq \begin{cases} M\, k^{1/2} & & \text{if } L_g < \lambda \\ M\, k^{\gamma/2}, & \text{with } \gamma = \frac{\lambda}{L_g} & \text{if } L_g > \lambda \\ M\, k^{\gamma/2}, & \gamma \in (0,1) & \text{if } L_g = \lambda \end{cases} \tag{60}$$

**Remark 3.2:** Estimate (60) indicates which is the best result it can be expected for this methodology of approximation, because estimate (60) is optimal as it is clearly seen looking at the counterexample given in §4.

**Remark 3.3:** In [5], [11], [12], [29], several errors estimates have been discussed, in [12], M. Falcone has established the estimate:

$$\left| u(x) - u_k^h(x) \right| \leq C \left(h + \frac{k}{h}\right)$$

In the usual case $h \sim k$ this estimate is useless, because convergence cannot be derived from it. Indeed, the real phenomenon of convergence is completely described by the estimation (60) and the counterexample given in §4.

b) Regular case

Here, the estimate (43) holds. Now we consider the minimization in h, in the differents cases we obtain:

• Case $\lambda > 2\,L_g$

We obtain that $\bar{h} = \frac{1}{2} k^{2/3}$ realizes the minimum; then the associated estimation becomes:

$$\left| u(x) - u_k^h(x) \right| \leq M\, k^{2/3}$$

• Case $L_g < \lambda < 2\,L_g$

We obtain that $\bar{h} = \frac{1}{\gamma}\, k^{2/(\gamma+1)}$ realizes the minimum, in this case the estimation is

$$\left| u(x) - u_k^h(x) \right| \leq M\, k^{\gamma/(\gamma+1)}$$

• Case $L_g > \lambda$

We obtain that $\bar{h} = k$ realizes the minimum, in this case the estimation is:

$$\left| u(x) - u_k^h(x) \right| \leq M\, k^{\gamma/2}$$

Making a summary we obtain:

$$\left| u(x) - u_k^h(x) \right| \leq \begin{cases} M\, k^{2/3} & \text{if } 2\,L_g < \lambda \\ M\, k^{\gamma/(\gamma+1)} & \text{if } L_g < \lambda < 2\,L_g \\ M\, k^{\gamma/2} & \text{if } L_g > \lambda \end{cases} \tag{61}$$

**Remark 3.4:** The estimates (61) indicate a continuous behavior of the maximum error corresponding to this type of discretizations. Also, there is a continuous variation of the point that realizes the minimum. This phenomenon is illustrated in Figure 1 and 2.

### 3.4.3 *A problematic case $h << k$*

When $h << k$ the estimate obtained in (42) although true, is useless and one might wonder if in this case convergence of the method is lost. Fortunately, this is not true and what is really happening with the convergence of solution $u_k^h$ is shown by the equivalence given in the following lines. In a few words, we will prove that in this case h can be replaced by $\hat{h}$ with $\hat{h} \sim k$, without any harmful effects because the following inequality is valid:

$$\left\| u_k^h - u_k^{\tilde h} \right\| \le \left(h + \tilde h + | h - \tilde h |\right) M_f \tag{62}$$

To see this, we are going to use the following estimate:

$$\left\| \hat u_k^h - \hat u_k^{\tilde h} \right\| \le M_f \, | h - \tilde h | \tag{63}$$

where $\hat u_k^h$ is the unique solution of the problem $\hat P_k$.

Let b be as in Figure 3. By virtue of (28) there exists $m_1$ *such that for any* $h \le m_1 k$, *for any vertex* $x_i$ and for any control $a \in A$, the point $x_i + h\, g(x_i, a)$ belongs to a neighboring simplex $S_j^k$.

We will suppose in the rest of this section that the following inequality holds:

$$h \le m_1 k \tag{64}$$

So

$$\hat u_k^h(x) = \min_{a \in A}\left( \frac{1}{1+\lambda h}\, \hat u_k^h(x + h\, g(x,a)) + h f(x,a)\right) =$$

$$= \min_{a \in A}\left( \frac{1}{1+\lambda h} \left(1 - \frac{\| g(x,a) \| h}{d}\, \hat u_k^h(x) + \frac{h \| g(x,a) \|}{d}\, \hat u_k^h(b) \right) + h f(x,a)\right) \tag{65}$$

with $d = \| x - b \|$, then

$$\hat u_k^h(x) \le \frac{\| g(x, \bar a_1) \|}{d \lambda + \| g(x, \bar a_1) \|}\, \hat u_k^h(b) + \frac{(1 + \lambda h) d}{\lambda d + \| g(x, \bar a_1) \|}\, f(x, \bar a_1) \tag{66}$$

where $\bar a_1$ readizes the minimum in (65), for $h = \tilde h$. Hence

$$\hat u_k^h(x) - \hat u_k^{\tilde h}(x) \le \beta\left( \hat u_k^h(b) - \hat u_k^{\tilde h}(b)\right) + \left( \frac{(1 + \lambda h) d}{\lambda d + \| g(x, \bar a_1) \|} - \frac{(1 + \lambda \tilde h) d}{\lambda d + \| g(x. \bar a_1) \|}\right) f(x, \bar a_1) \tag{67}$$

with

$$\beta = \frac{\| g(x, \bar a_1) \|}{d \lambda + \| g(x, \bar a_1) \|}$$

Analogously, we can estimate $\hat{u}_k^{\tilde{h}}(x) - \hat{u}_k^h(x)$ with the same bound appearing in (67) and then, we have:

$$\left\| \hat{u}_k^h - \hat{u}_k^{\tilde{h}} \right\| \leq M_f \ | h - \tilde{h} |$$

By (38) and (63), we obtain:

$$\left\| u_k^h - u_k^{\tilde{h}} \right\| \leq \left\| u_k^h - \hat{u}_k^h \right\| + \left\| \hat{u}_k^h - \hat{u}_k^{\tilde{h}} \right\| + \left\| \hat{u}_k^{\tilde{h}} - u_k^{\tilde{h}} \right\| \leq M_f \left( h + \tilde{h} + |h - \tilde{h}| \right)$$

From this inequality arises the above mentioned equivalence (62).

### 3.4.4 *Study of an asymptotic case*

We may consider the problem $\hat{P}_k$ and let the parameter h tend to zero. In that case we will arrive to the following fixed point problem

$$w_k(x) = \min_{a \in A} \left( \frac{\| g(x,a) \|}{\lambda d + \| g(x,a) \|} w_k(b) + \frac{d}{\lambda d + \| g(x,a) \|} f(x,a) \right) \tag{68}$$

where $d = \| b - x \|$, and b is the point described in the previous section and depicted in Figure 3.

This expression of the fully discrete problem was introduced in [17], where the convergence of the discrete solution $w_k$ to the continuous solution u was proved for the case of Lipschitz continuous solution (for which the heavy assumption $\lambda > L_g$ is necessary). As a simple by-product of the results of this paper we can obtain, for the expression (68) of the fully discrete problem, the convergence of solutions $w_k$ in the general case ( where $\lambda$ is arbitrary and u is only Hölder continuous ).

By (65) we have

$$\hat{u}_k^h(x) \leq \frac{\| g(x,\bar{a}) \|}{\lambda d + \| g(x,\bar{a}) \|} \hat{u}_k^h(b) + \frac{(1 + \lambda h) d}{\lambda d + \| g(x,\bar{a}) \|} f(x,\bar{a})$$

where $\bar{a}$ realizes the minimum in (68).

Then, we have:

$$\hat{u}_k^h(x) - w_k(x) \leq \beta \left( \hat{u}_k^h(b) - w_k(b) \right) + \left( \frac{(1 + \lambda h) d}{\lambda d + \| g \|} - \frac{d}{\lambda d + \| g \|} \right) f(x,\bar{a})$$

where

$$\beta = \frac{\| g(x,\bar{a}) \|}{\lambda d + \| g(x,\bar{a}) \|}$$

Then

$$\hat{u}_k^h(x) - w_k(x) \le \beta \left\| \hat{u}_k^h - w_k \right\| + \frac{\lambda h d}{\lambda d + \| g(x,\bar{a}) \|} M_f$$

Analogously, we can obtain the estimate:

$$w_k(x) - \hat{u}_k^h(x) \le \hat{\beta} \left\| \hat{u}_k^h - w_k \right\| + \frac{\lambda h d}{\lambda d + \| g(x,\hat{a}) \|} M_f$$

where

$$\hat{\beta} = \frac{\| g(x,\hat{a}) \|}{\lambda d + \| g(x,\hat{a}) \|}$$

and $\hat{a}$ realizes the minimum in (65).

By simple algebraic manipulations we arrive to the estimate:

$$\left\| \hat{u}_k^h - w_k \right\| \le M_f h \tag{69}$$

Now if we let $h \to 0$, we obtain $\hat{u}_k^h \to w_k$.

Finally, if u is the viscosity solution of problem (5), then by (38) , (42) and (69), we have:

$$\left\| u - w_k \right\| \le \left\| u - u_k^h \right\| + \left\| u_k^h - \hat{u}_k^h \right\| + \left\| \hat{u}_k^h - w_k \right\| \le (\sqrt{h} + \frac{k}{\sqrt{h}})^{\gamma} + 2 h M_f$$

This last inequality implies the convergence of solutions $w_k$ for the set-up (68) of the discrete problem. Also, an explicit estimate of the rate of convergence can be obtained, because by virtue of (64), if we take $h = m_1 k$ in the above expression, we get

$$\left\| u - w_k \right\| \le C k^{\gamma/2} \tag{70}$$

## 4. OPTIMALITY OF ESTIMATIONS $k^{1/2}$, $k^{2/3}$ and $k^{\gamma/2}$

### 4.1 Optimality of Estimate $k^{1/2}$

Let us consider a problem with the following characteristics:

The domain where the system moves is the circular ring of extremal radii $\bar{r}_1$, $\bar{r}_3$. Let $(\theta, r)$ be the polar coordinates of a general point x, i.e. $x=(\theta,r)$, $\theta\in[-\pi,\pi]$ and $\bar{r}_1 \le r \le \bar{r}_3$. The evolution of the system is described by the equations:

$$\dot{\theta} = 0$$

$$\dot{r} = \rho(r,a)$$

i.e. the system follows a radial movement; in consecuence, we have for g written in polar coordinates:

$$g(\theta,r,a) = (0,\ \rho(r,a))'$$

For $\bar{r}_1>0$, $\bar{r}_2=2\bar{r}_1$, $\bar{r}_3=2\bar{r}_2$, $p<1$, we define $A=\{a_1,a_2\}$ and:

$$\left|\begin{array}{l} \rho(r,a_1)=0 \quad \forall\ -\pi \le \theta \le \pi, \quad \bar{r}_1 \le r \le \bar{r}_3 \\ f(\theta,r,a_1)= \lambda(|\theta|+1)\,(\,|r-r_2|+1\,)\ \forall\ -\pi \le \theta \le \pi, \quad \bar{r}_1 \le r \le \bar{r}_3, \end{array}\right.$$

$$\rho(r,a_2) = \left|\begin{array}{ll} 1 & \forall\ -\pi \le \theta \le \pi, \quad \bar{r}_1 \le r \le \bar{r}_2 \\ -\frac{1}{2r_1} r + 2 & \forall\ -\pi \le \theta \le \pi, \quad \bar{r}_2 \le r \le \bar{r}_3 \end{array}\right.$$

$$f(\theta,r,a_2)= \left|\begin{array}{ll} \lambda\,(|\theta|+1) & \forall\ -\pi \le \theta \le \pi, \quad \bar{r}_1 \le r \le \bar{r}_2 \\ \lambda(|\theta|+1)\,(5\,(r-r_2)+1\,) & \forall\ -\pi \le \theta \le \pi, \quad \bar{r}_2 \le r \le \bar{r}_3 \end{array}\right. \tag{71}$$

It is easy to see that

$$u(0,\bar{r}_1) = 1$$

and that the optimal feedback control is $a_2$ for $\bar{r}_1 \le r \le \bar{r}_2$ and $a_1$ for $\bar{r}_2 \le r \le \bar{r}_3$.

We want to compute the discretized solution, so, we consider a triangulation of $\Omega$ of the type shown in

Figure 4, with $\Omega$ a circular ring of radii $\bar{r}_1$ and $\bar{r}_3$. Let $p \in \mathcal{N}^+$, for the chosen **triangulation the** following relation holds:

$$n = p \tag{72}$$

$$\alpha = \frac{2\pi}{p}$$

$$k = 2\, r_3 \sin(\frac{\pi}{p}) \tag{73}$$

$$\hat{h}_\nu = \frac{h - r_\nu(\cos(\alpha/2) - 1)}{\cos(\alpha/2)}$$

$$r_{\nu+1} = r_\nu + \hat{h}_\nu \qquad \nu = 0,1, \ldots, n-1 \quad r_0 = \bar{r}_1$$

$$h \text{ such that } \sum_{q=0}^{n-1} \hat{h}_q = \bar{r}_2 - \bar{r}_1 \tag{74}$$

For $\bar{r}_1 \leq r \leq \bar{r}_2$, we will identify the nodes of the triangulation with a pair of indices $\nu$, $\mu$

$$0 \leq \nu \leq n \tag{75}$$

$$-\frac{p}{2} \leq \mu \leq \frac{p}{2} \tag{76}$$

Denoting with $x(\nu,\mu)$ the node associated to the pair $(\nu,\mu)$, where if $x(\nu,\mu) = (\theta,r)$ it is

$$\theta = \mu\alpha - \frac{\alpha}{2}\,(\nu - 2\,[\tfrac{\nu}{2}])$$

$$r = r_\nu$$

For simplicity of writing we will use in this section the notation $w = u_k^h$.

For the values of $(\nu,\mu)$ given by (75) and (76), w can be compute recursively as we will do in the following lines. The key point of the development is the fact that:

$$x(\nu,\mu) + g(x(\nu,\mu),a_2)\,h = \frac{1}{2}\Big(x(\nu+1,\mu) + x(\nu+1,\mu+1)\Big)$$

Let $x(0, 0) = (0, r_1)$ , then by (29):

$$w(x(0,0)) = \Big((1-\lambda h)\ w(x(0,0)+h\ g(x(0,0),a_2))+h\ f(x(0,0),a_2)\Big)$$

but

$$w(x(0,0)+h\ g(x(0,0),a_2)) = \frac{1}{2}\ w(x(1,0)) + \frac{1}{2}\ w(x(1,1))$$

then the above formula becomes:

$$w(x(0,0)) = (1-\lambda h)\ \Big(\frac{1}{2}\ w(x(1,0)) + \frac{1}{2}\ w(x(1,1))\Big) + h\ f(x(0,0),a_2)$$

Duplicating this procedure and replacing $w(x(1,0))$ y $w(x(1,1))$ in formula (29), we obtain

$$w(x(0,0)) = (1-\lambda h)^2 \left(\tfrac{1}{2}\right)^2 \Big(w(x(2,-1))+2w(x(2,0))+w(x(2,1))\Big) +$$
$$+ h\ f(x(0,0)a_2) + h\ \frac{1-\lambda h}{2}\ \Big(f((x(1,0)),a_2) + f((x(1,1)),a_2)\Big) \qquad (77)$$

considering that the minimum of (71) is realized at $\theta=0$ and has the value $\lambda$, we obtain:

$$w(x(0,0)) \geq (1-\lambda h)^2 \left(\tfrac{1}{2}\right)^2 \Big(w(x(2,-1))+2w(x(2,0))+w(x(2,1))\Big) + h\lambda(1 + (1-\lambda h)$$

After n steps

$$w(x(0,0)) \geq (1-\lambda h)^n \left(\tfrac{1}{2}\right)^n \sum_{j=0}^{n} \binom{n}{j}\ w(x(n,j-[\tfrac{n}{2}])) + h\lambda \sum_{i=1}^{n} (1-\lambda h)^{i-1} \qquad (78)$$

where

$$\| x(n,j-[\tfrac{n}{2}]) \| = r_2$$

but when $\|x\|=r_2$

$$w(x) = (1-\lambda h)\ w(x+h\,g(x,a_1))+h\,f(x,a_1) = (1-\lambda h)\ w(x) + h\ f(x,a_1)$$

and then

$$w(x) = \frac{f(x,a_1)}{\lambda} = |\theta|+1$$

by formula (78), we have:

$$w(x(0,0)) \geq (1-\lambda h)^n \left(\tfrac{1}{2}\right)^n \sum_{j=0}^{n} \binom{n}{j} \Big(\Big|j - \tfrac{1}{2}\ n\Big|\alpha+1\Big) + h\lambda \sum_{i=1}^{n} (1-\lambda h)^{i-1} \qquad (79)$$

by Stirling's inequality, we have

$$\sqrt{2\pi}\, n^{n+\frac{1}{2}}\, e^{-n}\, e^{\frac{1}{12n+1}} < n! < \sqrt{2\pi}\, n^{n+\frac{1}{2}}\, e^{-n}\, e^{\frac{1}{12n}}$$

we can estimate the individual terms $\binom{n}{j}\left(\frac{1}{2}\right)^n$ because we know that:

$$\binom{n}{j}\left(\frac{1}{2}\right)^n \geq \binom{n}{\frac{n}{2}+\sqrt{n}}\left(\frac{1}{2}\right)^n \qquad \forall\, j \;/\; \frac{n}{2}-\sqrt{n} \leq j \leq \frac{n}{2}+\sqrt{n}$$

assuming $\sqrt{n}$ integer, we have

$$\frac{n!}{(\frac{n}{2}+\sqrt{n})!\,(\frac{n}{2}-\sqrt{n})!} > \frac{1}{\sqrt{2\pi}}\, e^{\eta}\, (\tfrac{1}{2})^{-n-1}\, \frac{1}{\sqrt{n}}\, (1-\tfrac{4}{n})^{-\frac{n}{2}-\frac{1}{2}}\, (\phi(n))^{-\sqrt{n}}$$

where

$$\eta = \frac{1}{12n+1} - \frac{1}{6n+12\sqrt{n}} - \frac{1}{6n-12\sqrt{n}}$$

and

$$\phi(n) = \frac{(\frac{n}{2}-2\sqrt{n})}{(\frac{n}{2}+2\sqrt{n})}$$

but:

$$e^{\eta} \rightarrow 1, \text{ when } n \rightarrow \infty$$

$$(1-\tfrac{4}{n})^{-\frac{n}{2}-\frac{1}{2}} \rightarrow e^{2}, \text{ when } n \rightarrow \infty$$

$$(\phi(n))^{-\sqrt{n}} \rightarrow e^{-4}, \text{ when } n \rightarrow \infty$$

then:

$$\binom{n}{j}\left(\frac{1}{2}\right)^n \geq \frac{C}{\sqrt{n}} \qquad \forall\, j \;/\; \frac{n}{2}-\sqrt{n} \leq j \leq \frac{n}{2}+\sqrt{n}\,,$$

for a large enough n.

By virtue of this last inequality we can estimate (79) in the following way, for a large enough n, considering also that:

$$h\,\lambda \sum_{i=1}^{n} (1-\lambda h)^{i-1} \rightarrow 1 - e^{-\lambda T}$$

and, $T = \bar{r}_2 - \bar{r}_1 = n\,h$

$$(1-\lambda h)^n \rightarrow e^{-\lambda T}, \text{ i.e. } (1-\lambda h)^n - e^{-\lambda T} \geq -Ch$$

$$w(x(0,0)) \geq (1-\lambda h)^n \left(\tfrac{1}{2}\right)^n \sum_{j=0}^{n} \binom{n}{j} \left(\left|j - \tfrac{1}{2}\, n\right| \alpha + 1\right) + h\lambda \sum_{i=1}^{n} (1-\lambda h)^{i-1} \geq$$

$$\geq e^{-\lambda T} \frac{C}{\sqrt{n}} \left( \sum_{j=\xi_1}^{\xi_2} \left|j - \tfrac{1}{2} n\right| \alpha \right) + e^{-\lambda T} \sum_{j=0}^{n} \binom{n}{j} \left(\tfrac{1}{2}\right)^n + 1 - e^{-\lambda T} - Ch$$

where

$$\xi_1 = \frac{n}{2} - \sqrt{n} \quad \text{and} \quad \xi_2 = \frac{n}{2} + \sqrt{n}, \text{ then:}$$

$$w(x(0,0)) \geq e^{-\lambda T} \frac{C}{\sqrt{n}} \alpha \left( \sum_{j=0}^{\sqrt{n}} j \right) + 1 - Ch \geq C \frac{1}{\sqrt{n}} + 1 - Ch \geq C_1 \sqrt{k} + 1$$

for a large enough n, by virtue of (74), (72) and (73). Hence

$$\max_{x \in \Omega} \left| u_k^h(x) - u(x) \right| \geq C_1 \sqrt{k}$$

and we see that the estimate (60) is optimal for this type of approximations.

### 4.2 Optimality of Estimate $k^{2/3}$

As it is shown in Figure 5, we consider the example above, but now each point $x_i$ is related with the point $x_i + h\, g(x_i, a)$, where the new time step h is of order $k^{2/3}$, i.e., estrictly

$$h = (\bar{r}_2 - \bar{r}_1)/p^{1/3}), \quad k = 2\bar{r}_3 \sin(\pi/(p+1)),$$

which implies

$$\lim_{p \to \infty} (h/k^{2/3}) = \bar{r}_1{}^{1/3}/(4\pi^{2/3})$$

Following the same procedure we obtain:

$$w(x(0,0)) = (1-\lambda h) \left( \tfrac{1}{2} w(x(p^{1/3},0)) + \tfrac{1}{2} w(x(p^{1/3},1)) \right) + h\, f(x(0,0), a_2)$$

Duplicating this procedure we obtain an equation similar to (78), and in parallel with the same estimate above, we have (we suppose $p^{1/3}$ integer):

$$w(x(0,0)) \geq (1-\lambda h)^m \left(\tfrac{1}{2}\right)^m \sum_{j=0}^{m} \binom{m}{j} w(x(m\,p^{1/3} j - [\tfrac{m}{2}])) + h\lambda \sum_{i=1}^{m} (1-\lambda h)^{i-1}$$

when $m = \bar{m} = p^{2/3}$, we are at $r = r_2$ and then we have:

$$w(x(0,0)) \geq C_1 \frac{1}{\sqrt{\bar{m}}} \bar{m}\, \alpha + 1 = C_2\, k^{2/3} + 1$$

hence

$$\max_{x \in A} \left| u_k^h(x) - u(x) \right| > C_2\, k^{2/3}$$

So, we see in this example that not only the procedure of approximation can be improved in the regular case (where (61) holds) but, that actually this improvement is the best one we can expect to obtain.

### 4.3 Optimality of Estimate $k^{\gamma/2}$

In this example, we consider that the domain $\Omega$ where the system moves is the circular ring of extremal radii $\bar{r}_1$, $\bar{r}_3$. If $(\theta, r)$ are the polar coordinate of a general point x,

$$x = (\theta, r),\ \bar{r}_1 = 1,\ \ \bar{r}_2 = 2,\ \bar{r}_3 = 3,\ -\pi \leq \theta \leq \pi$$

the evolution of the system is described by the equations:

$$\dot{\theta} = 2\pi$$

$$\dot{r} = \rho(r,a)$$

in consecuence, we have for g written in polar coordinates:

$$g(\theta,r,a) = (2\pi,\ \rho(r,a))'$$

We define:

$$\left|\begin{array}{lll} \rho(r,a_1) = 0 & \forall\ -\pi \leq \theta \leq \pi, & 1 \leq r \leq 3 \\ f(\theta,r,a_1) = \lambda & \forall\ -\pi \leq \theta \leq \pi, & 1 \leq r \leq 3, \end{array}\right.$$

$$\rho(r,a_2) = \left|\begin{array}{lll} r-2 & \forall\ -\pi \leq \theta \leq \pi, & 1.2 \leq r \leq 2.8 \\ 4(1-r) & \forall\ -\pi \leq \theta \leq \pi, & 1 \leq r \leq 1.2 \\ 4(3-r) & \forall\ -\pi \leq \theta \leq \pi, & 2.8 \leq r \leq 3 \end{array}\right.$$

$$f(\theta,r,a_2)=\begin{cases} 0 & \forall\ -\pi \le \theta \le \pi, \quad 1.5 \le r \le 2.5 \\ \frac{1.5-r}{0.3}\ \lambda & \forall\ -\pi \le \theta \le \pi, \quad 1 \le r \le 1.5 \\ \frac{r-2.5}{0.3}\ \lambda & \forall\ -\pi \le \theta \le \pi, \quad 2.5 \le r \le 3 \end{cases}$$

i.e. the system follows a spiral movement (in fact, at $r=\bar{r}_2$ a circular movement along the unstable limite cycle $r=\bar{r}_2$).

It is obvious that for this system, $L_g = 1$; so, to consider the case $\gamma = \lambda / L_g < 1$, we will suppose $\lambda < 1$.

It is easy to see that

$$u(\theta, \bar{r}_2) = 0, \ \forall\ \theta$$

and that the optimal control is:

$$a_2 \text{ for } 1.2 < r < 2.8 \text{ and } a_1 \text{ for } 1 \le r \le 1.2, \text{ or } 2.8 \le r \le 3$$

To compute the discretized solution, we consider the triangulation of $\Omega$ shown in Figure 6 (to depict more clearly the triangulation employed, in Figure 6 only a portion of $\Omega$ is shown). For $-n \le i \le n$, $0 \le j \le n$, we denote with x(i,j) the node associated to the pair (i,j). If $x(i,j) = (\theta,r)$,we have: ( being $nh = 1$ and $\sqrt{n}/2$ integer )

$$\begin{aligned}
\theta(i,j) &= j\,\frac{2\pi}{n} \\
r(i,j) &= 2 + i h \quad -\sqrt{n}/2 \le i \le \sqrt{n}/2, \text{ for j even} \\
r(i,j) &= 2 + (i-\tfrac{1}{2})h \quad 0 < i \le \sqrt{n}/2, \text{ for j odd} \\
r(i,j) &= 2 + (i+\tfrac{1}{2})h \quad -\sqrt{n}/2 \le i < 0, \text{ for j odd} \\
r(i,j) &= 2 + (i-\tfrac{1}{2})h \quad \sqrt{n}/2 < i \le n \quad \forall\ j \\
r(i,j) &= 2 + (i+\tfrac{1}{2})h \quad -n < i < -\sqrt{n}/2 \quad \forall\ j
\end{aligned}$$

From here results

$$k = h\sqrt{1+(6\pi)^2}$$

The proof of inequality:

$$\max_{x \in A} \left| u_k^h(x) - u(x) \right| > C_2\, k^{\gamma/2} \tag{80}$$

will be divided in two steps:

a) $w(i,j) \geq M\,|i\,h|^{\lambda}$, for all $i\ /\ \ n' \geq |i| > \frac{\sqrt{n}}{2}$, $n' = 0.8/h$ (81)

b) $w(0,j) \geq C\ w(\frac{\sqrt{n}}{2},j)$ (82)

a) As for $\sqrt{n}/2 < i \leq n'$ $\quad 0 \leq j \leq m$ , $f(x(i,j)) \geq 0$, the discrete equation (36) implies:

$$w(x(i,j)) \geq (1-\lambda h)\left( (\tfrac{1}{2} + r(i,j)-2)\ w(x(i+1,j+1)) + (\tfrac{1}{2} - r(i,j)+2)\ w(x(i,j+1)) \right) \tag{83}$$

with the obvious symmetry condition for j:

$$w(x(i,j)) = w(x(i,j')) \quad \forall\ j, j'$$

and the final condition

$$w(x(n',j)) = 1 \quad \forall\ j$$

If we define $\sigma(i,j) = \frac{1}{0.8^{\lambda}}\ (i\,h)^{\lambda}$, it follows that:

$$\sigma(i,j) = (1-\lambda h)\Big( (\tfrac{1}{2} + x(i,j))\,\sigma(i+1,j+1) + (\tfrac{1}{2} - x(i,j))\,\sigma(i-1,j+1) \Big) + \phi(h) \tag{84}$$

where $\phi(h) \leq C h^2$.

By (83) and (84), we have

$$w(i,j) - \sigma(i,j) \geq -\,C h$$

from where it easily follows (80) for $i>0$. Also, by reasons of symmetry, (80) is also valid for $i<0$.

b) To prove (82), we restrict the study of (11) to $-\frac{\sqrt{n}}{2} \leq i \leq \frac{\sqrt{n}}{2}$, in this case the equation (11) verified by w takes the form:

$$w(x(i,j)) = (1-\lambda h)\left( (\tfrac{1}{2} + r(i,j)-2)\, w(x(i+1,j+1)) + (\tfrac{1}{2} - r(i,j)+2)\, w(x(i-1,j+1)) \right)$$

with the boundary condition $w(x(\frac{\sqrt{n}}{2}+1,j))$, which has just been estimated in a).

The proof of (82) comprises three steps with the proofs of the properties described in (b1), (b2) and (b3).

b1)
$$w(x(i,j)) \leq w(x(i+1,j)) \quad i \geq 0$$

b2)
$$w(x(i,j) \geq \hat{w}(x(i,j) \quad -\frac{\sqrt{n}}{2} \leq i \leq \frac{\sqrt{n}}{2} \quad \forall j$$

where

$$\hat{w}(x(i,j)) = (1-\lambda h)\left( \tfrac{1}{2}\ \hat{w}(x(i+1,j+1)) + \tfrac{1}{2}\ \hat{w}(x(i\text{-}1,j+1)) \right)$$

with the boundary conditions :

$$\hat{w}(x(\sqrt{n}/2 + 1,j) = w(x(\sqrt{n}/2+1,j) \quad \forall j$$

$$\hat{w}(x(-\sqrt{n}/2 - 1,j) = w(x(-\sqrt{n}/2-1,j) \quad \forall j$$

Also, the following obvious symmetry condition for i holds:

$$w(x(-i,j)) = w(x(i,j))$$

and the symmetry condition for j:

$$w(x(i,j)) = w(x(i,j+2)) \tag{85}$$

where in (85) the addition is understood in the sense of mod(n).

b3)
$$\hat{w}(x(i,j) \geq \overline{w}(x(i,j) \quad -\frac{\sqrt{n}}{2} \leq i \leq \frac{\sqrt{n}}{2} \quad \forall j$$

where:

$$\overline{w}(x(i,j) = \left(1 - \frac{\lambda}{h} q^2 + \frac{\lambda}{h} (ih)^2\right) \xi$$

with $q=(\sqrt{n}/2+1)h$, and $\xi = \hat{w}(x(\frac{\sqrt{n}}{2}+1, j))$, then (for enough large n):

$$\overline{w}(x(0,j)) \geq \frac{1}{2}\ \hat{w}(x(\frac{\sqrt{n}}{2}+1, j)),$$

finally by (b2) and (b3) we obtain (82), because:

$$w(0,j) \geq C\ \hat{w}(0,j) \ \geq C\ \overline{w}(0,j) \ \geq \frac{C}{2}\ \hat{w}(\frac{\sqrt{n}}{2}+1,j) = \frac{C}{2}\ w(\frac{\sqrt{n}}{2}+1,j)$$

The proof of (b1), (b2) and (b3) are straightforward and will be omited here in order to simplify the presentation of the counterexample.

Finally, we arrive to the proof of (80), which is very easy taking into account (81) , (82) , also that $\gamma=\lambda$ (in this case $L_g=1$) and that

$$k = h\ \sqrt{1+(6\pi)^2}$$

By virtue of those mentioned inequalities we obtain:

$$w(0,j) \geq \frac{C}{2}\ w(\frac{\sqrt{n}}{2}+1,j)\ \geq M\ (\sqrt{n}\ h)^{\lambda} = M_1\ k^{\gamma/2}$$

Inequality which prove the optimality of estimate $k^{\gamma/2}$.

## CONCLUSIONS

This paper has analysed the problem of the error estimation in some usual discretization schemes of Hamilton-Jacobi equations corresponding to infinite horizon deterministic optimal control problems. We have given explicit error estimates and we have shown that they are optimal. The explicit estimates depend on the regularity of the data; in general, only an estimate of the type $\sqrt{k}$ is attainable, but in the regular case, where semiconcavity assumptions hold, estimate of type $k^{\gamma/(\gamma+1)}$ can be obtained for values of $\gamma \in (1,2)$ It should be remarked that some of the optimality results, which indicate the convenience of using an unusual large time step h in relation with the space discretization step k (in fact $h \sim k^{2/3}$) give rise to a new methodology of discretization and emphasize the harmful effect of the space discretization.


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## APPENDIX

### Proof of technical results.

### A.1 Proof of Lemma 3.1

Proof: Let $\{ a_p(\cdot) \, / \, p=0,1,... \} \subset \mathcal{A}$ be a minimizing sequence, i.e.

$$\inf_{a\in\mathcal{A}} \int_0^T f(y(t,a(t)))e^{-\lambda t}\, dt = \lim_{p\to\infty} \int_0^T f(y(t,a_p(t)))e^{-\lambda t}\, dt$$

then:

$$u(x) - u_T(x) = \inf_{a\in\mathcal{A}} \int_0^\infty f(y(t),a(t))e^{-\lambda t}\, dt - \inf_{a\in\mathcal{A}} \int_0^T f(y(t),a(t))e^{-\lambda t}\, dt \leq$$

$$\leq \int_0^T f(y(t),a_p(t))e^{-\lambda t}\, dt + \frac{M_f}{\lambda} e^{-\lambda T} - \lim_{p\to\infty} \int_0^T f(y(t),a_p(t))e^{-\lambda t}\, dt \quad \forall\, p$$

Then

$$u(x) - u_T(x) \leq \frac{M_f}{\lambda} e^{-\lambda T}$$

In a similar way, we can obtain:

$$u_T(x) - u(x) \leq \inf_{a\in\mathcal{A}} \int_0^T f(y(t,a(t)))e^{-\lambda t} dt - \lim_{p\to\infty} \int_0^T f(y(t,a_p(t)))e^{-\lambda t} dt + \frac{M_f}{\lambda} e^{-\lambda T} \leq \frac{M_f}{\lambda} e^{-\lambda T}$$

By virtue of these inequalities

$$\left| u(x) - u_T(x) \right| \leq \frac{M_f}{\lambda} e^{-\lambda T}$$

□

### A.2 Proof of Lemma 3.3

Proof: $A_k^h$ is a contractive operator, with fixed point $u_k^h$, i.e.

$$u_k^h = A_k^h \, u_k^h,$$

also, $u_{k,T}^h$ is defined recursively by:

$$u^h_{k,T} = (A^h_k)^{\mu}(w_T),$$

being $w_T = 0$. So, by (35) we have:

$$\left| u^h_{k,T}(x) - u^h_k(x) \right| \le (1-\lambda h)^{\mu} \left| u^h_k \right|,$$

By (17), we have:

$$\left| u^h_{k,T}(x) - u^h_k(x) \right| \le \frac{M_f}{\lambda} (1-\lambda h)^{\mu} \le \frac{M_f}{\lambda} e^{-\lambda T} \tag{86}$$

□

### A.3 Preliminaries for the proof of Lemma 3.4

To prove Lemma 3.4 we need some auxiliary results.

#### A.3.1 *Regularity of* $u^h_T$

**Proposition 1:** $u^h_T$ *is Lipschitz continuous and moreover*

$$L_{u^h_T} \le \begin{cases} L_f \dfrac{1}{L_g - \lambda} e^{(L_g-\lambda)T} & \textit{if } L_g > \lambda \\ L_f \dfrac{1}{\lambda - L_g} & \textit{if } L_g < \lambda \\ L_f \, T & \textit{if } L_g = \lambda \end{cases}$$

Proof: It is clear that for $n = \mu$, $u^h_\mu = 0$ is Lipschitz continuous and $L_{u^h_\mu} = 0$

To complete the induction procedure, we consider the following inequality:

$$u^h_n(x) - u^h_n(y) \le (1-\lambda h)\, u^h_{n+1}(x+h\,g(x,\tilde a)) + h\,f(x,\tilde a) - (1-\lambda h)\, u^h_{n+1}(y+h\,g(y,\tilde a)) + h\,f(y,\tilde a)$$

where $\tilde a$ realizes the minimum of (9) for $u^h_n(y)$, then

$$u^h_n(x) - u^h_n(y) \le \Big((1-\lambda h)(1+L_g h)\, L_{u^h_{n+1}} + L_f\, h\Big) \| x - y \|$$

Analogously for $u^h_n(y) - u^h_n(x)$, then we obtain:

$$L_{u_n^h} \leq (1-\lambda h)(1+L_g h)\, L_{u_{n+1}^h} + L_f\, h \tag{87}$$

Analysis of different cases

• $L_g = \lambda$

In this case we have that $(1-\lambda h)(1+L_g h) = 1-(\lambda h)^2 < 1$, then:

$$L_{u_n^h} \leq L_{u_{n+1}^h} + L_f\, h$$

which implies

$$L_{u_0^h} \leq L_{u_\mu^h} + \mu\, L_f\, h = \mu h\, L_f = L_f T$$

• $L_g < \lambda$

The formula (87) becomes

$$L_{u_n^h} \leq (\, 1 - (\lambda - L_g)h\,)\, L_{u_{n+1}^h} + L_f\, h$$

then

$$L_{u_n^h} \leq L_f h\, \frac{1 - (\, 1 - (\lambda - L_g)h\,)^n}{h\,(\lambda - L_g)} \leq L_f\, \frac{1}{\lambda - L_g}$$

• $L_g > \lambda$

The inequality (87) becomes

$$L_{u_n^h} \leq L_f h\, \frac{(\, 1 + (L_g - \lambda)h\,)^n}{h\,(L_g - \lambda)}$$

so

$$L_{u_T^h} \leq L_f\, \frac{1}{L_g - \lambda}\, e^{(L_g - \lambda)T}$$

□

**Remark A.3.1:** From here we will denote with $L_u$ the Lipschitz constant of $u_T^h$ and we will discriminate the three cases between them when it were necessary.

A.3.2 *Definition of* $u^h_{n\rho}$, *regularization of* $u^h_n$

By convolution with a smooth function $\beta(\cdot)$, we obtain a regular approximation of $u^h_n$

$$u^h_{n\rho}(x) = \left(u^h_n * \beta_\rho\right)(x) = \int_{B(\rho)} u^h_n(x-y)\, \beta_\rho(y)\, dy \tag{88}$$

where

$$\beta_1(\cdot) \in C^\infty(\Re^\nu), \tag{89}$$

$$\beta_1(x) \geq 0 \ \forall x, \text{ support of } \beta_1 \subset B_1 = \{\, x \in \Re^\nu \,/\, \| x \| \leq 1 \,\} \tag{90}$$

$$\int_{\Re^\nu} \beta_1(x)\, dx = 1 \tag{91}$$

$$\beta_\rho(x) = \frac{1}{\rho^\nu}\, \beta_1(\tfrac{x}{\rho}) \geq 0 \quad \forall \rho \in \Re^+ \tag{92}$$

A.3.3 *Definition of* $\tilde{u}^h_{n\rho}$, *linear interpolation of* $u^h_{n\rho}$

We define $\tilde{u}^h_{n\rho}$ the element of $W_k$ such that:

$$\tilde{u}^h_{n\rho}(x_i) = u^h_{n\rho}(x_i), \ \forall\ i = 1, N \tag{93}$$

**Proposition 2:** *Let* g,f *be functions such that* $u^h_n$ *is Lipschitz continuous, then:*

$$\left| u^h_n(x) - u^h_{n,\rho}(x) \right| \leq L_u\, \rho \tag{94}$$

Proof: By definition of convolution we have:

$$\left| u^h_n(x) - u^h_{n,\rho}(x) \right| = \left| u^h_n(x) - \int_{B(\rho)} u^h_n(x-y)\, \beta_\rho(y)\, dy \right| =$$

$$= \int_{B(\rho)} \left| u^h_n(x) - u^h_n(x-y) \right| \beta_\rho(y)\, dy$$

but, being $u^h_n$ Lipschitz continuous, it holds

$$\left| u_n^h(x) - u_{n,\rho}^h(x) \right| \leq L_u \int_{B(\rho)} \| y \| \, \beta_\rho(y) \, dy \leq L_u \, \rho \qquad \square$$

**Proposition 3:** *Functions* $u_n^h$, $u_{n,\rho}^h$ *and operator* $A_a^h$ *verify inequalities* (95) *and* (96)

$$u_n^h(x) \leq \left( A_a^h u_{n+1}^h \right)(x) \quad \forall\, a \in A \tag{95}$$

$$u_{n,\rho}^h(x) \leq \left( A_a^h u_{n+1}^h * \beta_\rho \right)(x) \quad \forall\, a \in A \tag{96}$$

*also* $\forall$ *Lipschitz continuous function* w, *we have:*

$$\left| (A_a^h \; w) * \beta - A_a^h \; (w*\beta) \right| \leq (1-\lambda h) \, L_w L_g \, \rho \, h + L_f \, \rho \, h \tag{97}$$

*in addition*

$$\left| \left( A_a^h \; u_{n,\rho}^h \right)(x) - \left( A_a^h \; \tilde{u}_{n,\rho}^h \right)(x) \right| \leq C(1-\lambda h) L_u \frac{k^2}{\rho} \tag{98}$$

Proof: (95) and (96) are trivial, for (97) we have:

$$\left( (A_a^h \; w) * \beta - A_a^h \; (w*\beta) \right)(x) = \int_{B(\rho)} \left( (1-\lambda h) \, w(x-\eta+h\,g(x-\eta,a)) + h\,f(x-\eta,a) \right) \beta_\rho(\eta) \, d\eta \, -$$

$$-(1-\lambda h) \int_{B(\rho)} w(x-\eta+h\,g(x,a)) \, \beta_\rho(\eta) \, d\eta - h\,f(x,a)$$

then

$$\left| \left( (A_a^h \, w) * \beta - A_a^h \, (w*\beta) \right)(x) \right| \leq \int_{B(\rho)} (1-\lambda h) \left| w(x-\eta+h\,g(x-\eta,a) - w(x-\eta+h\,g(x,a)) \right| \beta_\rho(\eta) d\eta \, +$$

$$+ \, h \int_{B(\rho)} \left| f(x-\eta,a) - f(x,a) \right| \beta_\rho(\eta) \, d\eta$$

$$\leq (1-\lambda h) \, L_w L_g \, h \, \rho + L_f \, h \, \rho$$

and (97) is proved.

Function $u^h_{n,\rho}$ has second derivatives bounded by

$$\left\| D^2 u^h_{n,\rho} \right\| \le \tilde{C}\, \frac{L_u}{\rho} \tag{99}$$

where $\tilde{C}$ is a constant depending only of $\beta_1(\cdot)$, because it is the regularization (by the function $\beta_\rho$) of the Lipschitz continuous function $u^h_n$ and by virtue of (89)$-$(92). In consequence by virtue of condition (27) and (99) the difference between $u^h_{n,\rho}$ and its linear interpolation $\tilde{u}^h_{n,\rho}$ is bounded by $C\, L_u \frac{k^2}{\rho}$ (for a detailed proof of these properties see [17], [28]),and then we have:

$$\left| \left(A^h_a\, u^h_{n,\rho}\right)(x) - \left(A^h_a\, \tilde{u}^h_{n,\rho}\right)(x) \right| \le (1-\lambda h) \left| ( u^h_{n,\rho} - \tilde{u}^h_{n,\rho})(x + g(x,a)h) \right| \le C\,(1-\lambda h)\, L_u \frac{k^2}{\rho}$$

and (98) is proved.

□

Proposition 4: *The following estimate holds*

$$\tilde{u}^h_{T\rho}(x) - u^h_{kT}(x) \le \frac{1-\lambda h}{\lambda}\, L_u\, L_g\, \rho + \frac{L_f}{\lambda}\, \rho + \frac{1-\lambda h}{\lambda h}\, C_1\, \frac{L_u\, k^2}{\rho} \tag{100}$$

Proof: We define:

$$E_n = \sup_{x \in \Omega_k} \left( \tilde{u}^h_{n\rho}(x) - u^h_{kn}(x) \right) = \max_{i=1,\dots N} \left( \tilde{u}^h_{n\rho}(x_i) - u^h_{kn}(x_i) \right) \tag{101}$$

By properties (96) and (97) we have:

$$u^h_{n-1,\rho}(x) = \left(A^h u^h_n * \beta_\rho\right)(x) \le \left(A^h_a u^h_n * \beta_\rho\right)(x) \le (A^h_a\, (u^h_n * \beta))\,(x) + \phi(x,h,\rho) =$$

$$= (A^h_a\, u^h_{n,\rho})\,(x) + \phi(x,h,\rho) \tag{102}$$

with

$$\left|\phi(x,h,\rho)\right| \le (1-\lambda h)\, L_u\, L_g\, h\, \rho + L_f\, h\, \rho$$

By (34) and (48), if $w \in W^k$, then:

$$\left(A^h_k\, w\right)(x) = \min_{a \in A} \left((1-\lambda h)\, w(x+h\, g(x,a))+h\, f(x,a)\right) = \left(A^h\, \tilde{w}\right)(x) \le \left(A^h_a\, \tilde{w}\right)(x) \tag{103}$$

then, by (103), (102) and (98) we have:

$$u^h_{n-1,\rho}(x_i) \leq \left(A^h_a \ \tilde{u}^h_{n,\rho}\right)(x_i) + \phi(x_i,h,\rho) + \psi(x_i,h,\rho.k) \leq$$

$$\leq \left(A^h_a \ \tilde{u}^h_{n,\rho}\right)(x_i) + (1-\lambda h)\, L_w L_g h \rho + L_f \, h \rho + C(1-\lambda h) L_u \frac{k^2}{\rho}$$

so

$$u^h_{n-1,\rho}(x_i) \leq \left(A^h_a \ \tilde{u}^h_{n,\rho}\right)(x_i) + \phi(x_i,h,\rho) + \psi(x_i,h,\rho.k) \quad \forall a$$

and

$$u^h_{k,n-1}(x_i) = \left(A^h_k \ u^h_{k,n}\right)(x_i) = \left(A^h_{\bar{a}} \ u^h_{k,n}\right)(x_i)$$

where $\bar{a}$ realizes the minimum in (103) for $u^h_{k,n}$, then:

$$u^h_{n-1,\rho}(x_i) - u^h_{k,n-1}((x_i) \leq \left(A^h_{\bar{a}} \ \tilde{u}^h_{n,\rho}\right)((x_i) + \phi(x_i,h,\rho) + \psi(x_i,h,\rho.k) - \left(A^h_{\bar{a}} \ u^h_{k,n}\right)(x_i)$$

By (93), we have

$$\tilde{u}^h_{n-1,\rho}(x_i) - u^h_{k,n-1}(x_i) \leq$$

$$\leq (1-\lambda h)\ (\ \tilde{u}^h_{n,\rho}(x_i) - u^h_{k,n}(x_i)) + (1-\lambda h)\, L_u L_g h \rho + L_f \, h \rho + C(1-\lambda h) L_u \frac{k^2}{\rho}$$

so, by definition (101)

$$E_{n-1} \leq (1-\lambda h)\, E_n + (1-\lambda h)\, L_u L_g h \rho + L_f \, h \rho + C(1-\lambda h) L_u \frac{k^2}{\rho} \tag{104}$$

As $E_{n-1} \leq \beta E_n + b$ implies $E_0 \leq \beta^n E_n + \frac{b}{1-\beta}$, then replacing this estimate in (104) and taking into account that $E_\mu = 0$, we obtain (100).

□

Proposition 5: *The following estimate holds:*

$$u^h_{kT}(x) - \tilde{u}^h_{T\rho}(x) \leq \frac{1-\lambda h}{\lambda} L_u \ L_g \ \rho + \frac{L_f}{\lambda} \rho + \frac{1-\lambda h}{\lambda h} C_1 \frac{L_u k^2}{\rho} + 2 L_u \rho \tag{105}$$

Proof: Let $\hat{x}_j$ be an arbitrary vertex of $\Omega_k$, then there exist $\bar{a}_1(\hat{x}_j)$, ..., $\bar{a}_\mu((\hat{x}_j)$ such that:

$$u^h_T(\hat{x}_j) = u^h_0(\hat{x}_j) = (A^h_{\bar{a}_1} u^h_1)(\hat{x}_j) = ((A^h_{\bar{a}_1} A^h_{\bar{a}_2} \ldots A^h_{\bar{a}_\mu}) u^h_\mu)(\hat{x}_j) \tag{106}$$

where $\bar{a}_n$ realizes the minimum that appears in the definion of operator $A^h$ applied to function $u_n^h$ and computed at the point defined recursively by

$$y^h(\hat{x}_j, n) = y^h(\hat{x}_j, n\text{-}1) + h\, g(y^h(\hat{x}_j, n\text{-}1), \bar{a}_n))$$

with initial condition

$$y^h(\hat{x}_j, 0) = \hat{x}_j$$

We define $v_n^h$ in the following way

$$v_n^h = A^h_{\bar{a}_{n+1}} \; v^h_{n+1} \qquad \forall\; 0 \le n \le \mu\text{-}1$$

$$v_\mu^h = 0$$

Obviously, by (106), we have:

$$v_T^h(\hat{x}_j) = u_T^h(\hat{x}_j)$$

We define

$$\tilde{E}_n = \sup_{x \in \Omega_k} \left( u^h_{k,n}(x) - \tilde{v}^h_{n\rho}(x) \right) \tag{107}$$

Note: The construction of function $v_n^h$ depends on $\hat{x}_j$, but for simplicity of notation we will write it without denoting explicitly this dependence.

It is easy to prove that $v_n^h$ has the same regularity properties of $u_n^h$, i.e. it is Lipschitz continuous with the same Lipschitz constant and also it holds for it the estimates (97) and (98). Then, we have for any $x_i$ vertex of $\Omega_k$ that:

$$v^h_{n+1,\rho}(x_i) = \left(A^h_{\bar{a}_{n+1}} \; \tilde{v}^h_{n,\rho}\right)(x_i) + \phi(x_i,h,\rho) + \psi(x_i,h,\rho.k)$$

As

$$u^h_{k,n+1}(x_i) = \left(A^h u^h_{k,n}\right)(x_i) \le \left(A^h_{\bar{a}_{n+1}} \; u^h_{k,n}\right)(x_i),$$

we have:

$$u^h_{k,n+1}(x_i) - v^h_{n+1,\rho}(x_i) \le \left(A^h_{\bar{a}_{n+1}} u^h_{k,n}\right)(x_i) - \left(A^h_{\bar{a}_{n+1}} \tilde{v}^h_{n,\rho}\right)(x_i) + \left|\phi(x_i,h,\rho)\right| + \left|\psi(x_i,h,\rho.k)\right|$$

by the above inequality, (12) and (107), we have

$$\tilde{E}_{n+1} \le (1-\lambda h)\, \tilde{E}_n + (1-\lambda h)\, L_u\, L_g\, h\,\rho + h\, L_f\, \rho + C_1\, (1-\lambda h)\, \frac{L_u}{\rho}\, k^2$$

From here we obtain:

$$u^h_{kT}(x_i) - v^h_{T\rho}(x_i) \le \frac{1-\lambda h}{\lambda}\, L_u\, L_g\, \rho + \frac{L_f}{\lambda}\, \rho + \frac{1-\lambda h}{\lambda h}\, C_1\, \frac{L_u\, k^2}{\rho} \tag{108}$$

As for $n=\mu$ $\quad v^h_T(\hat{x}_j) = u^h_T(\hat{x}_j)$, we have:

$$\left| u^h_{T,\rho}(\hat{x}_j) - v^h_{T,\rho}(\hat{x}_j) \right| \le \left| u^h_{T,\rho}(\hat{x}_j) - u^h_T(\hat{x}_j) \right| + \left| u^h_T(\hat{x}_j) - v^h_T(\hat{x}_j) \right| + \left| v^h_T(\hat{x}_j) - v^h_{T,\rho}(\hat{x}_j) \right| \le 2\, L_u\, \rho$$

the inequality (108) becomes:

$$u^h_{kT}(\hat{x}_j) - u^h_{T\rho}(\hat{x}_j) \le \frac{1-\lambda h}{\lambda}\, L_u\, L_g\, \rho + \frac{L_f}{\lambda}\, \rho + \frac{1-\lambda h}{\lambda h}\, C_1\, \frac{L_u\, k^2}{\rho} + 2\, L_u\, \rho$$

Finally, as $\hat{x}_j$ is an arbitrary vertex of $\Omega_k$ (105) holds.

□

### A.4 Proof of Lemma 3.4

Proof: Obviously

$$\left| u^h_T(x) - u^h_{k,T}(x) \right| \le \left| u^h_T(x) - u^h_{T,\rho}(x) \right| + \left| u^h_{T,\rho}(x) - \tilde{u}^h_{T,\rho}(x) \right| + \left| \tilde{u}^h_{T,\rho}(x) - u^h_{k,T}(x) \right| \tag{109}$$

so, applying proposition 2, (100) y (105) we can estimate the right hand side of (109) and we have:

$$\left| u^h_T(x) - u^h_{k,T}(x) \right| \le L_u\, \rho + C\, \frac{L_u\, k^2}{\rho} + \frac{1}{\lambda}\, L_u\, L_g\, \rho + \frac{L_f}{\lambda}\, \rho + \frac{1}{\lambda h}\, C_1\, \frac{L_u\, k^2}{\rho} + 2\, L_u\, \rho \tag{110}$$

Analysis of different cases

- $L_g > \lambda$

In this case, $L_u$ takes the form:

$$L_u = L_f\, \frac{1}{L_g - \lambda}\, e^{(L_g-\lambda)T}$$

Inequality (110) becomes

$$\left| u^h_T(x) - u^h_{k,T}(x) \right| \le M\, e^{(L_g-\lambda)T} (\rho + \frac{k^2}{h\,\rho}) \tag{111}$$

where

$$M = \max\left( 3 + \frac{L_g}{\lambda} + L_g - \lambda\,,\ \frac{C}{\lambda} + \frac{C_1}{\lambda} \right) \frac{L_f}{L_g - \lambda}$$

Minimizing in $\rho$ the expression (111) $(\rho + \frac{k^2}{h\,\rho})$ we have the minimum is realized by $\rho = \frac{k}{\sqrt{h}}$, so:

$$\left| u^h_T(x) - u^h_{k,T}(x) \right| \le 2\, M\, e^{(L_g-\lambda)T} \frac{k}{\sqrt{h}}$$

• $L_g < \lambda$

in this case $L_u = \frac{L_f}{\lambda - L_g}$ and the inequality (110), becomes:

$$\left| u(x) - u^h_k(x) \right| \le M \left(\rho + \frac{k^2}{h\rho}\right)$$

where

$$M = \max\left(3 + \frac{L_g}{\lambda} + \lambda - L_g\,,\ \frac{C}{\lambda} + \frac{C_1}{\lambda}\right) \frac{L_f}{\lambda - L_g}$$

and in the same way that it has been done above, we have:

$$\left| u^h_T(x) - u^h_{k,T}(x) \right| \le 2\, M \frac{k}{\sqrt{h}}$$

• $L_g = \lambda$

In this case $L_u = L_f\, T$ and inequality (110) becomes

$$\left| u(x) - u^h_k(x) \right| \le M\, T \left( \rho + \frac{k^2}{h\rho} \right)$$

where

$$M = \max\left(4,\ \frac{C}{\lambda} + \frac{C_1}{\lambda}\right) L_f$$

then

$$\left| u^h_T(x) - u^h_{k,T}(x) \right| \le 2\, M\, T \frac{k}{\sqrt{h}}$$

□

## REFERENCES


[1] Aragone L. S., González, R. L., Tidball, M. M.: Solución numérica de juegos diferenciales de suma nula con controles monótonos. *Anales del Sèptimo Congreso Brasilero de Automàtica.* Vol 2, pp. 1078-1083, (Saò Jose Dos Campos, Brasil, 15-19 August 1988).

[2] Belbas S. A., Mayergoyz I. D.: Applications of fixed-point methods to discrete variational and quasi-variational inequalities. *Numerische Mathematik*, Vol. 51, pp. 631-654, 1987.

[3] Capuzzo Dolcetta I.: On a Discrete Approximation of the Hamilton-Jacobi Equation of Dynamic Programming. *Appl. Math. Optim.*, 10, pp. 367-377, 1983.

[4] Capuzzo Dolcetta I., Ishii H.: Approximate solution of the Bellman equation of deterministic control theory. *Appl. Math. Optim.*, Vol. 11, pp. 161-181, 1984.

[5] Capuzzo Dolcetta I., Falcone M.: Discrete dynamic programming and viscosity solutions of the Bellman equation, Preprint URLS-DM/NS-88/001, Roma, 1988.

[6] Ciarlet P.G.: Discrete maximum principle for finite-difference operators. *Aequations Math.*, 4, pp. 338-352, 1970.

[7] Ciarlet P.G., Raviart P. A.: Maximum principle and uniform convergence for the finite element method. *Computer Methods in Applied Mechanics and Engineering*, 2, pp. 17-31, 1973.

[8] Crandall M. G., Lions P. L.: Viscosity solutions of Hamilton-Jacobi equations. *Trans. AMS*, Vol. 277, pp. 1-42, 1983.

[9] Crandall M. G., Lions P. L.: Two approximation of solution Hamilton-Jacobi equations. *Mathematics of Computation*, Vol. 43, N°167, pp. 1-19, 1984.

[10] El Tarazi M. N.: On a monotony-preserving accelerator process for the successive approximations method. *IMA Journal of Numerical Analysis*, Vol. 6, pp. 439-446, 1986.

[11] Falcone M.: A Numerical Approach to the Infinite Horizon Problem of Deterministic Control Theory. *Appl. Math. Optim.*, 15, pp. 1-13, 1987

[12] Falcone M.: Corrigenda "A Numerical Approach to the Infinite Horizon Problem of Deterministic Control Theory". *Appl. Math. Optim.*, to appear, 1990.

[13] Fleming W. H., Rishel R. W.: *Deterministic and stochastic optimal control* (Springer-Verlag, Berlin-Heidelberg-New York, 1975).

[14] Friedman A. : *Differential Games* (Wiley-Interscience, New York, 1971).

[15] Glowinski R.: *Numerical Methods for Nonlinear Variational Problems* (Springer-Verlag, New York, 1984).

[16] Glowinski R., Lions J. L., Trémolières R.: *Numerical Analysis of Variational Inequalities* (North Holland, Amsterdam, 1981).

[17] González R., Rofman E.: On deterministic control problems: an approximation procedure for the optimal cost. Part I and II. *SIAM Journal on Control and Optimization*, 23, pp. 242-285, 1985.

[18] González R., Sagastizábal C.: Un algorithme pour la résolution rapide d'équations discrètes de Hamilton-Jacobi-Bellman. *Comptes Rendus Acad. Sc. Paris*, Serie I, Tome 311, pag. 45-50, 1990.

[19] González R., Tidball M.: Fast solution of discrete Isaacs's inequalities, *Rapport de Recherche* N° 1167, INRIA, 1990.

[20] González R., Tidball M.: On a Discrete Time Approximation of the Hamilton-Jacobi Equation of Dynamic Programming, *Rapport de Recherche*, INRIA, to appear, 1990.

[21] Lions P. L., Mercier B.: Approximation numérique des équations de Hamilton-Jacobi-Bellman. *R.A.I.R.O. Analyse numérique / Numerical Analysis,* Vol. 14, N°4, pp. 369-393, 1980.

[22] Lions P. L.: *Generalized solutions of Hamilton-Jacobi equations* (Pitman, London, 1982).

[23] Mosco U.: An introduction to the approximate solution of variational inequalities. In *Constructive Aspects of Functional Analysis*, Vol. II, pp. 497-682, (Edizioni Cremonese, Roma, 1971).

[24] Ortega J. M., Rheinboldt W. C.: *Iterative solution of nonlinear equations in several variables* (Academic Press, New York, 1970).

[25] Rockafellar T. R.: *Convex Analysis* (Princeton, New Jersey, 1970).

[26] Ross S. M.: *Applied Probability Models with Optimization Applications* (Holden-Day, San Francisco, 1970).

[27] Souganidis P. E.: Max-min representation and product formulas for the viscosity solution of Hamilton-Jacobi equations with application to differential games. *Nonlinear Analysis, Theory, Methods & Applications,* Vol. 9, N°3, pp. 217-257, 1985.

[28] Strang G., Fix G.: *An Analysis of the Finite Element Method* (Prentice-Hall, Englewood Cliffs, NJ, 1973).

[29] Tidball M. M.: Comments on "A Numerical Approach to the Infinite Horizon Problem of Deterministic Control Theory". *Appl. Math. Optim.,* to appear, 1990.

FIGURE 1

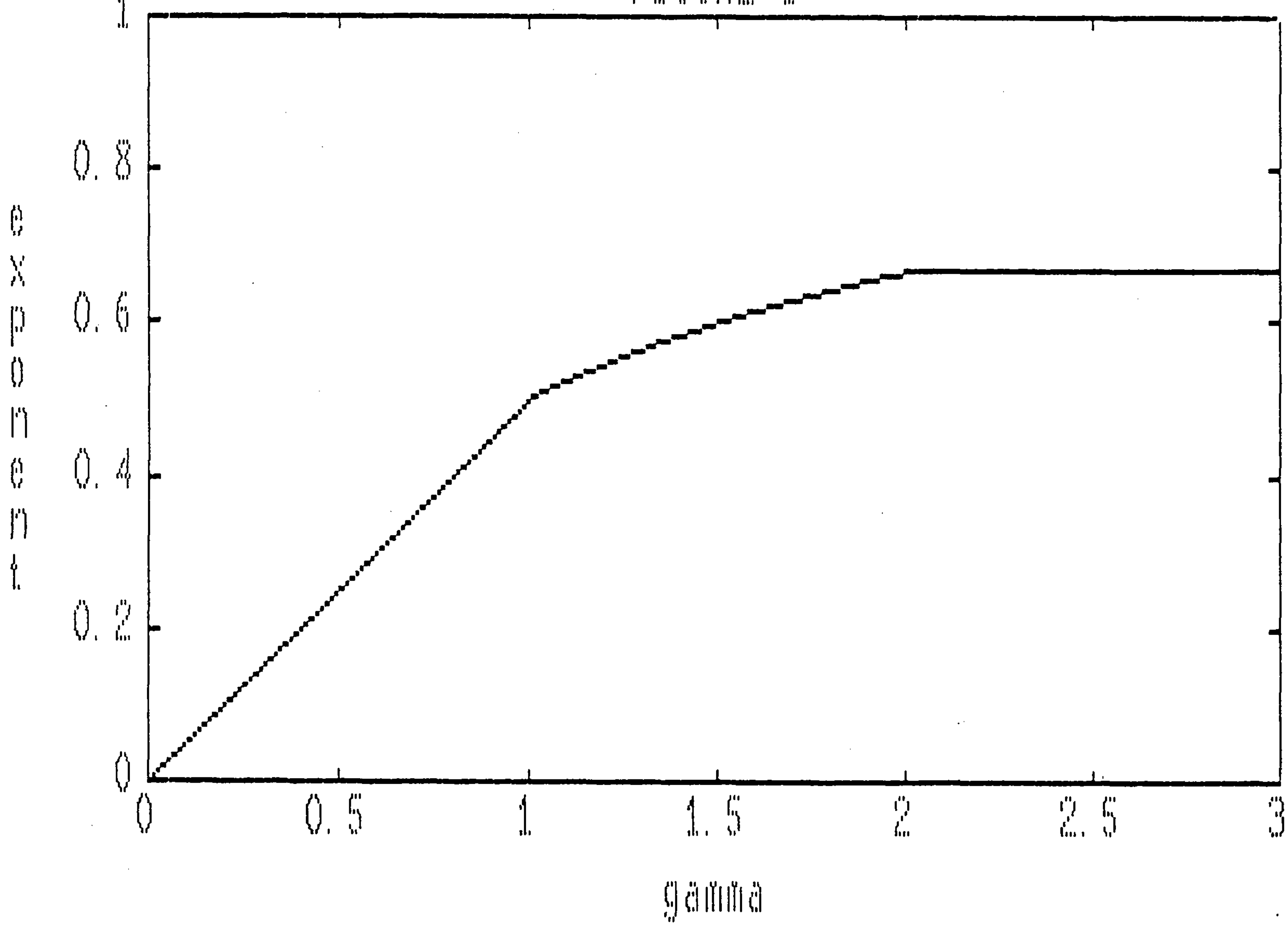


FIGURE 2

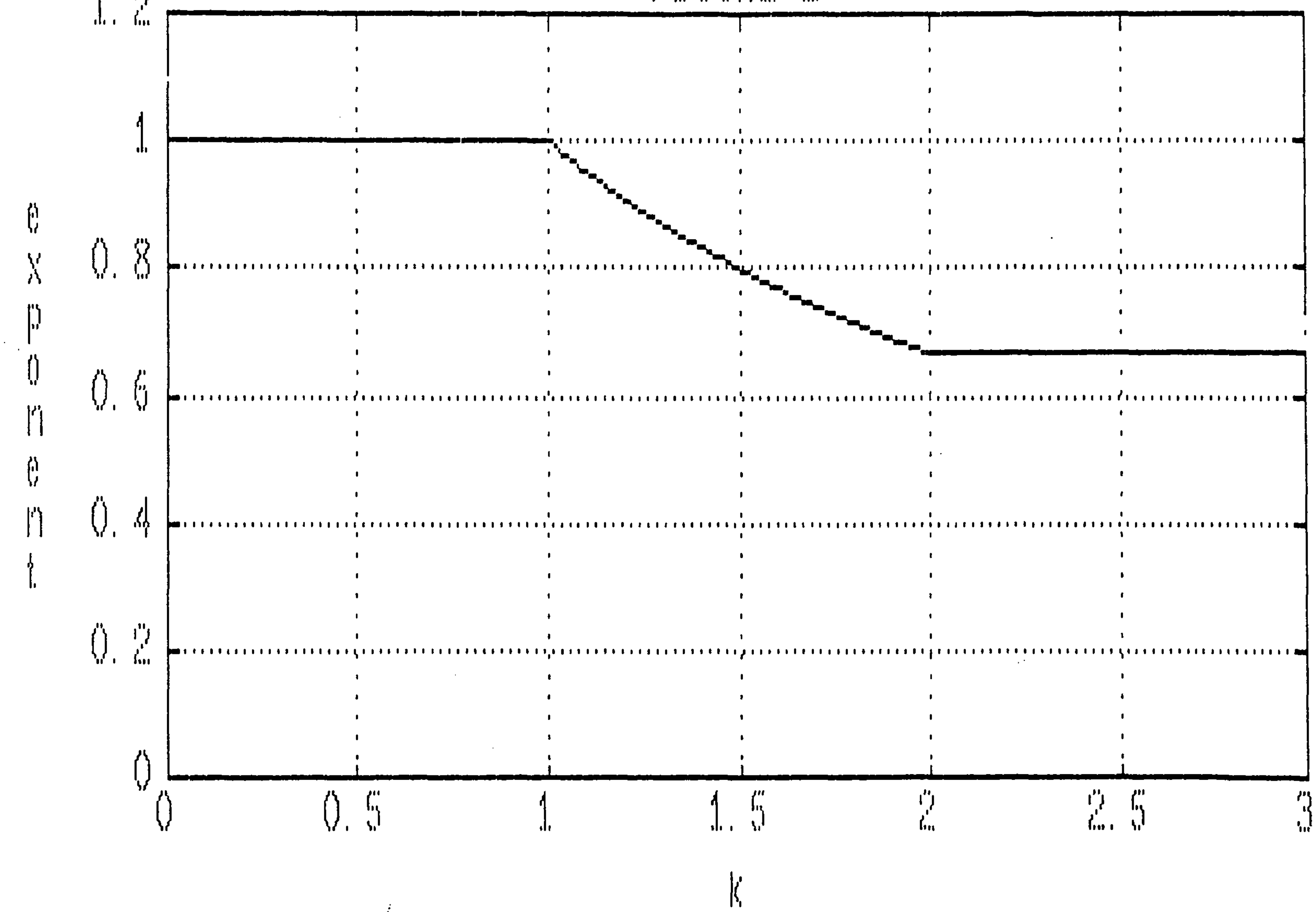


FIGURE 2

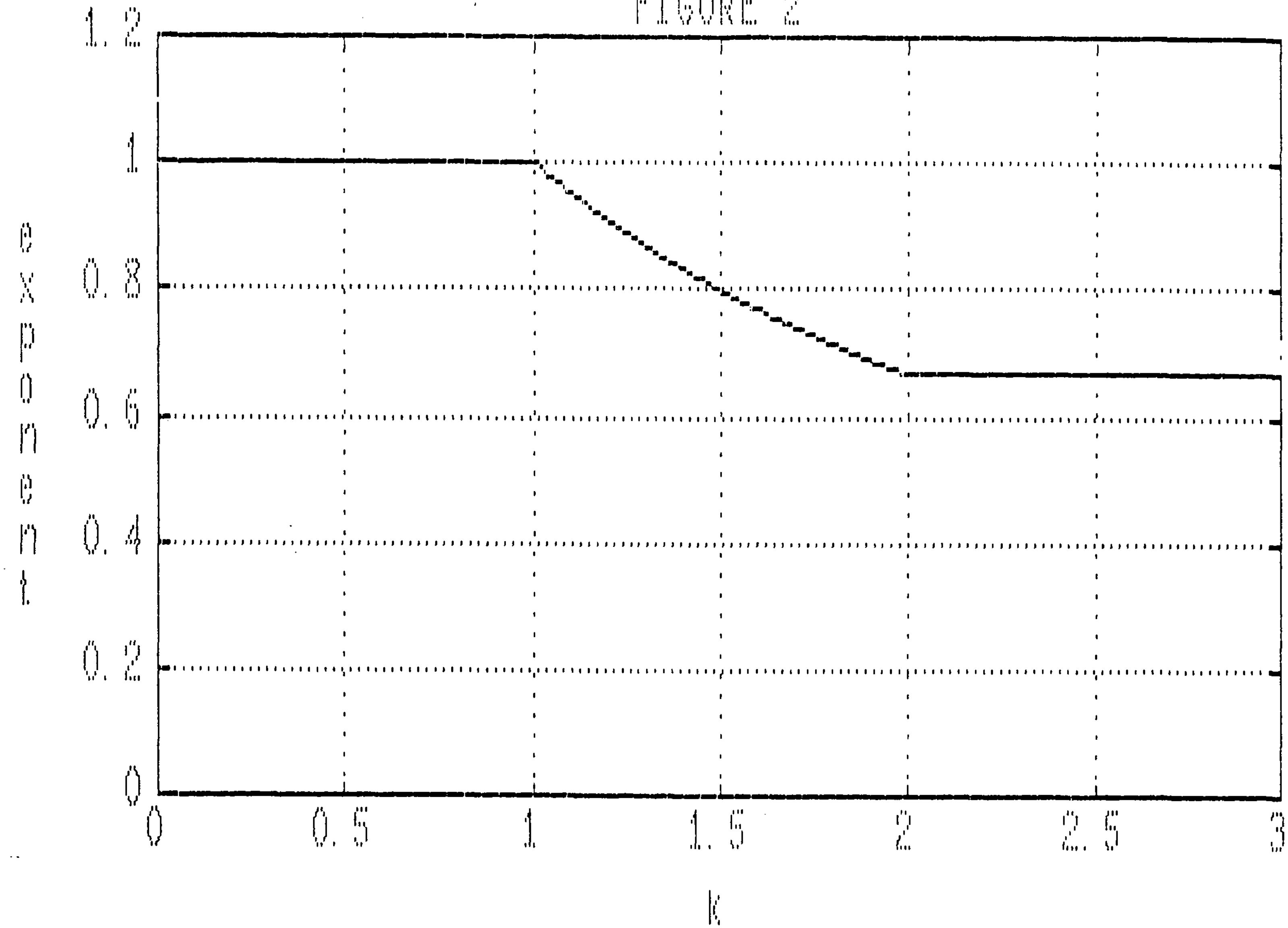
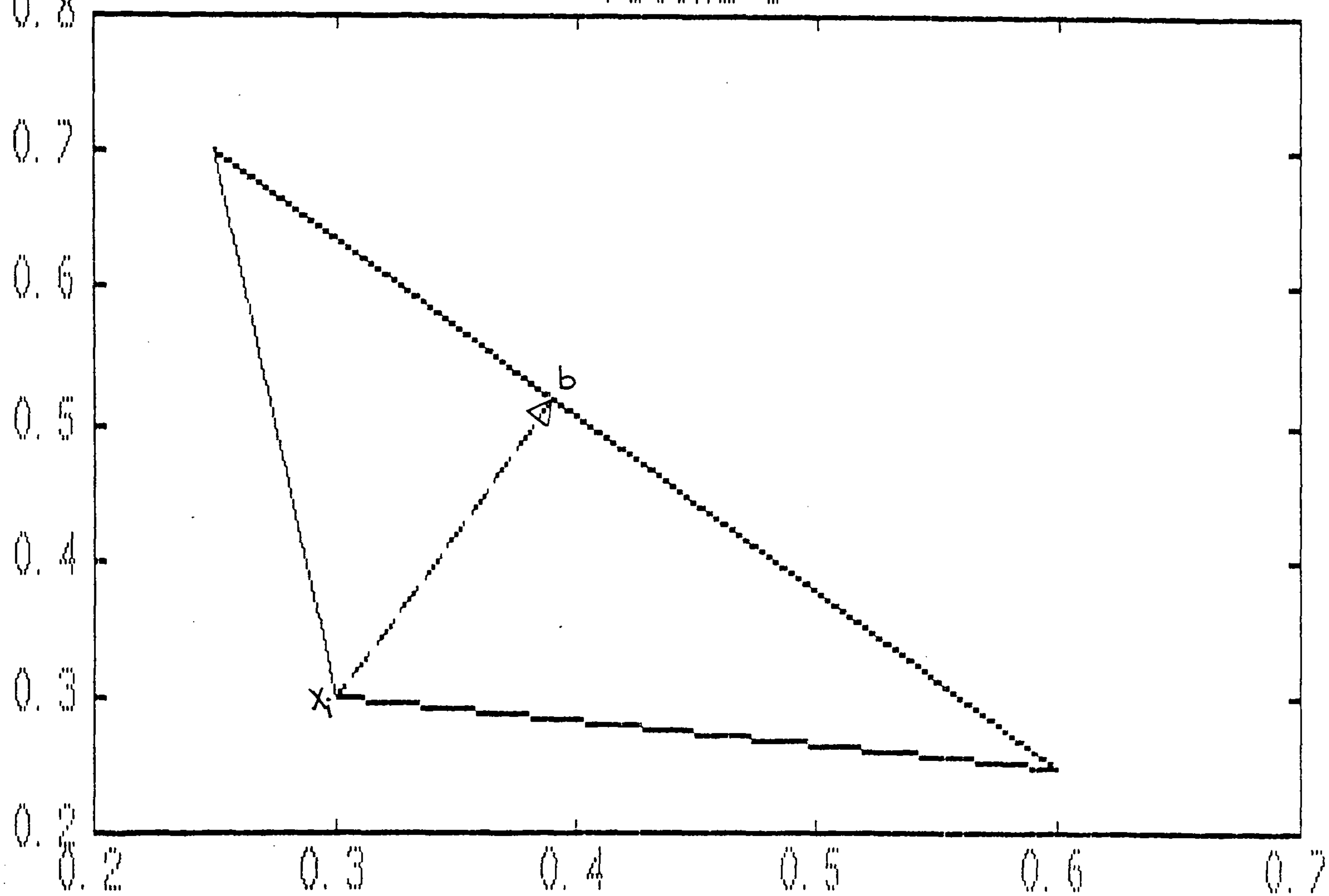


FIGURE 3



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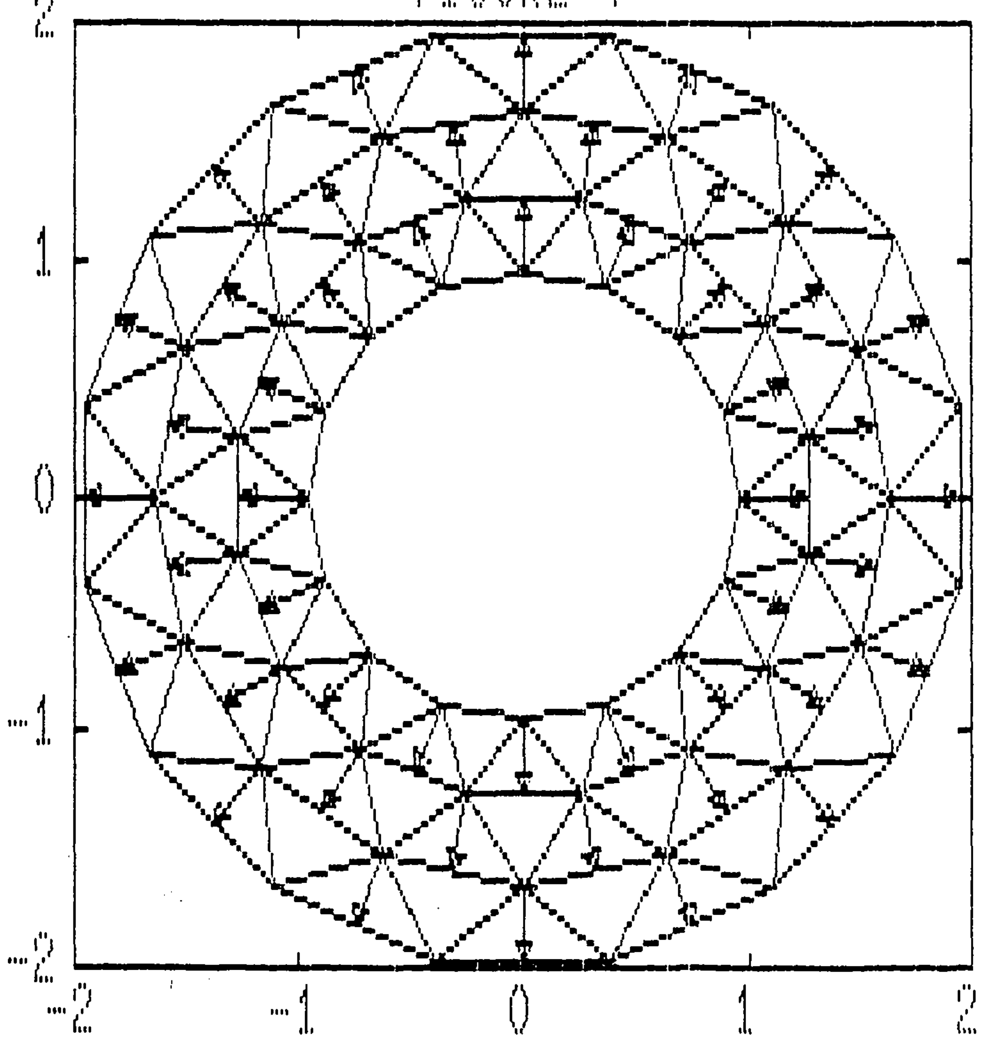

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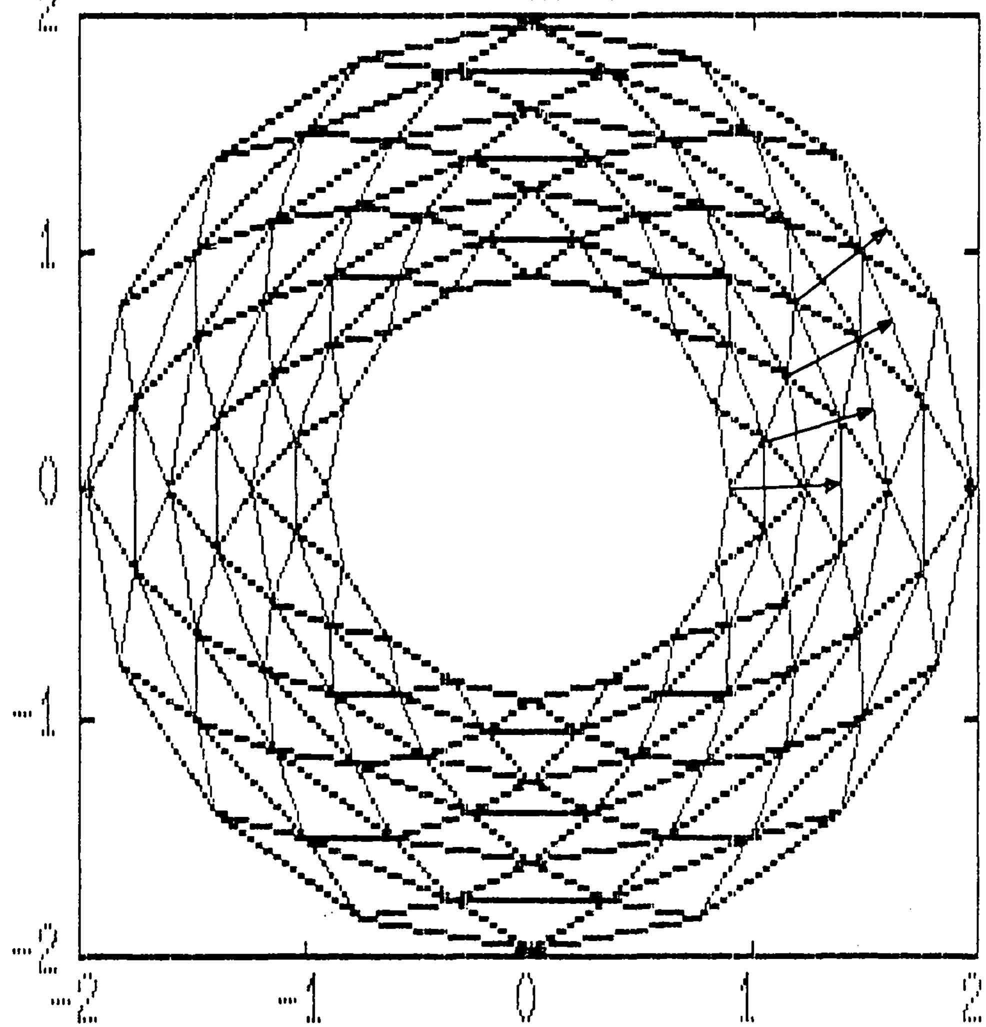

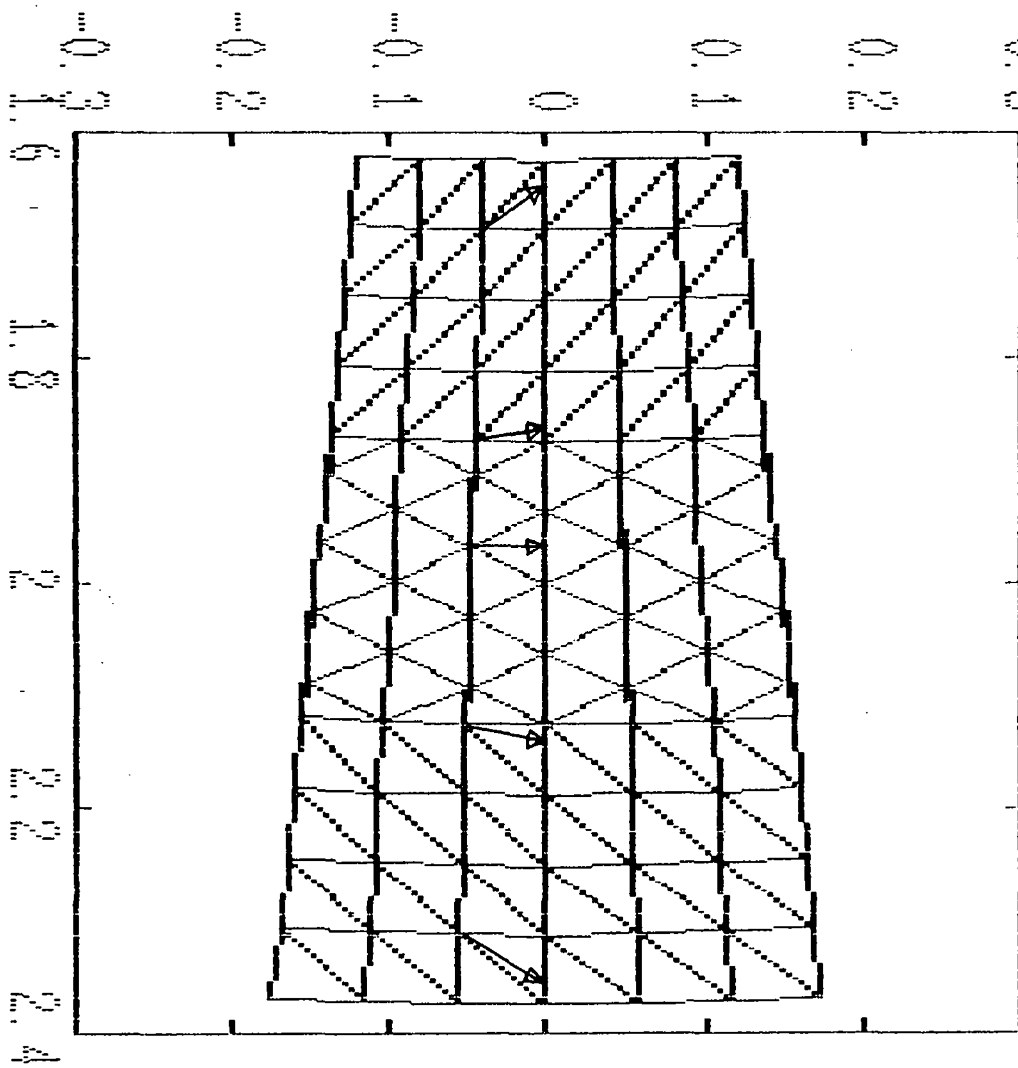

FIGURE 6

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